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Rationalized Haar Series for Approach Homogeneous and Non-Homogeneous State Equations

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Abstract: In this paper, we give a new approximate solution method to homogeneous and non-homogeneous state equations using rationalized Haar functions. Integration operational matrix of rationalized Haar functions are used to convert the computation of homogeneous and non homogeneous state equations to a simple system of algebraic equations. By using the method (based on Matlab programming) on numerical analysis examples, we show that our method has high degree of accuracy.

Keywords: Rationalized Haar functions, Homogeneous and non-homogeneous state equations, Integration Operational Matrix, Kronecker product.

متسلسلات هار النسبية لتقريب معادلات الحالة المتجانسة وغير المتجانسة

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كلية التربية المقداد || جامعة ديالى || العراق

المستخلص: في هذا البحث نعطي طريقة حل تقريبي جديدة لمعادلات الحالة المتجانسة وغير المتجانسة باستخدام دوال هار النسبية. يتم استخدام المصفوفة التشغيلية للتكامل لدوال هار النسبية لتحويل حساب معادلات الحالة المتجانسة وغير المتجانسة الى نظام بسيط من المعادلات الجبرية. باستخدام الطريقة (القائمة على برمجة الماتلاب) على أمثلة التحليل العددي نظهر أن طريقتنا تمتلك درجة عالية من الدقة.

الكلمات المفتاحية: دوال هار النسبية، معادلات الحالة المتجانسة وغير المتجانسة، مصفوفات العمليات تكامل، ضرب كرونكر.

1. Introduction.

In this paper we solve the state equations of linear time invariant (LTI) system [10] as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1r} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2r} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \quad \dots (1)$$

Where,

$\dot{\boldsymbol{x}}(t)$: is the vector function with n components $\boldsymbol{x_i}(t)$; $\boldsymbol{x_1}(t), \boldsymbol{x_2}(t), \boldsymbol{x_3}(t), \dots, \boldsymbol{x_n}(t)$: the state variables; $\boldsymbol{u_1}(t), \boldsymbol{u_2}(t), \boldsymbol{u_3}(t), \dots, \boldsymbol{u_r}(t)$: the inputs system; $\boldsymbol{A}$ is an $n \times n$ square matrix of the constant coefficients a_{ij} and $\boldsymbol{B}$ is an $n \times r$ matrix of the constant coefficients b_{ij} that weights the inputs.If $\boldsymbol{A}$ is a constant matrix and input control forces are zero then the eq. (1) takes the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \dots (2)$$

Such an equation is called homogeneous equation. The exact solution of eq. (2) is:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}\,(0)$$

If $\boldsymbol{A}$ is a constant matrix and matrix $\boldsymbol{u}(t)$ is non-zero vector then the equation takes normal form as,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

Such an equation is called non homogeneous equation. The exact solution of eq. (2) is:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

LTI system was studied, analyses, and solved in many authors based on the orthogonal functions, orthogonal polynomials and Fourier series, see [1, 3-5, 9, 12-13, 17]. Gu, Y and Zhu, Y[7], were used Haar wavelet transformation in optimal control problem to transform the proposed problem into an approximate uncertain optimal control problem with arbitrary accuracy because the dimension of Haar basis tends to infinity. In [14] Haar wavelet collocation method was used to obtain the numerical solution of linear Volterra and Fredholm integral equations. In this work we will give a general numerical solution for the homogeneous and non-homogeneous state equations. The new approach is much simpler in theory and more suitable for digital computation.

Ohkita M. and Kobayashi Y.[16] were first studied the rationalized Haar functions with their application in solution of linear differential equations. The rationalized Haar functions have been widely applied in: numerical technique for solving the classical brachistochrone problem in the calculus of variations[15], obtain eigenvalues of fractional Strum-Liouville problem [6], numerical solution of non-linear second kind two-dimensional integral equations[2], solve three dimensional non-linear Fredholm-integral equations[11].

Study problem:

The study problem can be formulated by the following questions:

1- Can the concept of rationalized Haar functions be used, studied and analysis to obtain the numerical general approach solution for solving homogeneous and non homogeneous stat equations?

2- It is possible to find the integral operational matrix of rationalized Haar functions in terms of integral operational matrix of Block pulse functions?

3- It can be able to generally this method into any orthogonal functions rather than rationalized Haar functions?

4- How to characterization the approximation solution via rationalized Haar functions?

Study supposal:

Our study in this paper will be related to define the rationalized Haar functions with their integral operational matrix, representation of any square function f (t) in terms of rationalized Haar series, as well as applied rationalized Haar series in compute the numerical solution of homogeneous and non homogeneous state equations. Finally some examples are considered to explain our method. The place and time of the study was in June of this year at the college of engineering – department of control system in university of Diyala.

Important of the study:

This work can be classified into three important of the study, first it can be used another representation method for computing inegrational operational rationalized Haar matrix based on Block pulses functions. Second it is presentation the numerical solution of homogeneous and non homogeneous state equations. Finally, the method shown that there are very close values between the exact solution and the approximation method.

Study methodology.

Our work composed of many information, results which related to our goal for instance discuses, study and analysis of rationalized Harr functions. The Matlab programming used to compute the approximation solution compared with the exact solution, this is shown in some examples in our paper.

2. Rationalized Haar Functions (RHF).

The rationalized Haar functions consists of the following functions, [16]:

$$\boldsymbol{RH}(0,t) = 1, \forall\, t, t \in [0,1)$$

$$\boldsymbol{RH}(1,t) = \begin{Bmatrix} 1 & t \in [0,1/2) \\ -1 & t \in [1/2,1) \\ 0 & o.w. \end{Bmatrix}, \boldsymbol{RH}(2,t) = \begin{Bmatrix} 1 & t \in [0,1/4) \\ -1 & t \in [1/4,1/2) \\ 0 & o.w. \end{Bmatrix},$$

$$\boldsymbol{RH}(3,t) = \begin{Bmatrix} 1 & t \in [1/2,3/4) \\ -1 & t \in [3/4,1) \\ 0 & o.w. \end{Bmatrix}, \boldsymbol{RH}(4,t) = \begin{Bmatrix} 1 & t \in [0,1/8) \\ -1 & t \in [1/8,1/4) \\ 0 & o.w. \end{Bmatrix},$$

$$\boldsymbol{RH}(5,t)=\begin{Bmatrix}1 & t\in[1/4,3/8)\\ -1 & t\in[3/8,1/2)\\ 0 & o.w.\end{Bmatrix},\boldsymbol{RH}(6,t)=\begin{Bmatrix}1 & t\in[1/2,5/8)\\ -1 & t\in[5/8,3/4)\\ 0 & o.w.\end{Bmatrix}$$

$$\boldsymbol{RH}(7,t)=\begin{Bmatrix}1 & t\in[3/4,7/8)\\ -1 & t\in[7/8,1)\\ 0 & o.w.\end{Bmatrix}$$

In general:

$$\boldsymbol{RH}(i,t)=\begin{Bmatrix}1 & t\in[I_1,I_{1/2})\\ -1 & t\in[I_{1/2},I_0)\\ 0 & o.w.\end{Bmatrix}$$

Where, $I_v=\frac{r-v}{2^n}, v=0,1/2,1$ and $i=2^n+r-1, n=0,1,2,...,r=1,2,...,2^n$.

Remark: Rationalized Haar functions are belong to class of complete orthonormal systems in Hilbert space $L^2[0,1]$, [16].

It can be assembled RHF as a square matrix of order m by dividing the closed interval $[0,1]$ into m subintervals with length $1/m$, where $m=2^l$, for some $l\in N=\{1,2,3,...\}$. We denote the collection points by: $t_s=\frac{2s-1}{2m}$, where $s=1,2,3,...,m$ and

$$\boldsymbol{RH}_c(t)=[\boldsymbol{RH}\left(c,\frac{1}{2m}\right)\boldsymbol{RH}\left(c,\frac{3}{2m}\right)\boldsymbol{RH}\left(c,\frac{2s-1}{2m}\right)\dots\boldsymbol{RH}\left(c,\frac{2m-1}{2m}\right)]$$

Where, $c=0,1,2,...,m-1$:

$$\boldsymbol{RH}_m(t)=\begin{bmatrix}\boldsymbol{RH}_0(t)\\ \boldsymbol{RH}_1(t)\\ \boldsymbol{RH}_2(t)\\ \vdots\\ \boldsymbol{RH}_{m-1}(t)\end{bmatrix}\quad\dots(3)$$

$$=\begin{bmatrix}\boldsymbol{RH}\left(0,\frac{1}{2m}\right) & \boldsymbol{RH}\left(0,\frac{3}{2m}\right) & \boldsymbol{RH}\left(0,\frac{2s-1}{2m}\right) & \dots & \boldsymbol{RH}\left(0,\frac{2m-1}{2m}\right)\\ \boldsymbol{RH}\left(1,\frac{1}{2m}\right) & \boldsymbol{RH}\left(1,\frac{3}{2m}\right) & \boldsymbol{RH}\left(1,\frac{2s-1}{2m}\right) & \dots & \boldsymbol{RH}\left(1,\frac{2m-1}{2m}\right)\\ \boldsymbol{RH}\left(2,\frac{1}{2m}\right) & \boldsymbol{RH}\left(2,\frac{3}{2m}\right) & \boldsymbol{RH}\left(2,\frac{2s-1}{2m}\right) & \dots & \boldsymbol{RH}\left(2,\frac{2m-1}{2m}\right)\\ \boldsymbol{RH}\left(3,\frac{1}{2m}\right) & \boldsymbol{RH}\left(3,\frac{3}{2m}\right) & \boldsymbol{RH}\left(3,\frac{2s-1}{2m}\right) & \dots & \boldsymbol{RH}\left(3,\frac{2m-1}{2m}\right)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \boldsymbol{RH}\left(m-1,\frac{1}{2m}\right) & \boldsymbol{RH}\left(m-1,\frac{3}{2m}\right) & \boldsymbol{RH}\left(m-1,\frac{2s-1}{2m}\right) & \dots & \boldsymbol{RH}\left(m-1,\frac{2m-1}{2m}\right)\end{bmatrix}$$

Where, $\boldsymbol{RH}_m(t)$ is called the rational Haar matrix of order $m = 2^l$, for some $l \in N$ and $\boldsymbol{RH}_0(t), \boldsymbol{RH}_1(t), \ldots, \boldsymbol{RH}_{m-1}(t)$ are the rationalized Haar vectors. For example, consider the rationalized Haar matrices of order 2, 4, and 8 respectively:

$$\boldsymbol{RH}_2(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{RH}_4(t) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\boldsymbol{RH}_8(t) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Definition (2.2): A spectrum function $f\ (t)$ over the interval [0, 1), can be expanded in a rationalized Haar functions series with an infinite number of terms [16]:

$$f(t) = \sum_{i=0}^{\infty} \alpha_i \boldsymbol{RH}\ (i, t) \quad \ldots (4)$$

Where, $\boldsymbol{RH}\ (i, t)$ are the rationalized Haar functions and the rationalized Haar coefficients can be written as:

$$\alpha_i = \frac{1}{\gamma_i} \int_0^1 f(t) \boldsymbol{RH}(i, t) dt \quad \ldots (5)$$

Where, $\gamma_i = \int_0^1 RH(i, t) RH(i, t) dt = 2^{-n}$ are called the normalized factors for RHF's.

Definition (2.3): The n-th degree rationalized Haar functions approximation of an arbitrary function $f(t)$ over [0, 1] is represented by

$$f(t) = \sum_{i=0}^{m-1} \alpha_i \boldsymbol{RH}(i, t) = \boldsymbol{\alpha}^{\boldsymbol{T}}\ \boldsymbol{RH}_m\ (t) \quad \ldots (6)$$

Where, $\boldsymbol{\alpha}^{\boldsymbol{T}} = [\ \alpha_0\ \alpha_1\ \alpha_2 \ldots \alpha_{m-1}]$ is called the coefficients vector and $\boldsymbol{RH}_m(t) = [\boldsymbol{RH}_0(t)\ \boldsymbol{RH}_1(t)\ \boldsymbol{RH}_2(t) \ldots \boldsymbol{RH}_{m-1}(t)]$ is the rationalized Haar vector.

3. Integration Operational Matrix For RHF's.

The approximation of the integral of a rationalized Haar vector $\boldsymbol{RH}_m(t)$ in eq. (3) can be represented mathematically as:

$$\int_0^t \boldsymbol{RH}_m(x)dx \approx \boldsymbol{E}_m \boldsymbol{RH}_m(t) \quad \ldots (7)$$

Where, $\boldsymbol{E}_m$ is called a $m \times m$ square operational matrix of integration uniquely determined by $\boldsymbol{RH}_c(t)$, Where, $c = 0, 1, 2, \ldots, m-1$, which is given as:

$$\boldsymbol{E}_m = \boldsymbol{RH}_m(t)\boldsymbol{E}_m^* (\boldsymbol{RH}_m(t))^{-1} \quad \ldots (8)$$

Where,

$$\boldsymbol{E}_m^* = \frac{1}{m}\begin{bmatrix} \frac{1}{2} & 1 & 1 \ldots 1 \\ 0 & \frac{1}{2} & 1 \ldots 1 \\ 0 & 0 & \frac{1}{2} \ldots 1 \\ \vdots & \vdots & \vdots \ldots \vdots \\ 0 & 0 & 0 \ldots 1/2 \end{bmatrix} \quad \ldots (9)$$

$\boldsymbol{E}_m^*$ is called the operational square matrix for Block pulse functions, [18]. It can be noted that eq. (8) was computed as Wu, [18] sense in integration operational matrix for orthogonal bases vector based on $\boldsymbol{E}_m^*$. For example, consider the operational matrices for rationalized Haar functions of order 4 and 8 respectively:

$$\boldsymbol{E}_4 = \begin{bmatrix} 0.5000 & -0.2500 & -0.1250 & -0.1250 \\ 0.2500 & 0.0000 & -0.1250 & 0.1250 \\ 0.0625 & 0.0625 & 0.0000 & 0.0000 \end{bmatrix}$$

$$\boldsymbol{E}_8 = \begin{bmatrix} 0.5000 & -0.2500 & -0.1250 & -0.1250 & -0.0625 & -0.0625 & -0.0625 & -0.0625 \\ 0.2500 & 0.0000 & -0.1250 & 0.1250 & -0.0625 & -0.0625 & 0.0625 & 0.0625 \\ 0.0625 & 0.0625 & 0.0000 & 0.0000 & -0.0625 & 0.0625 & 0.0000 & 0.0000 \\ 0.0625 & -0.0625 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0625 & 0.0625 \\ 0.0156 & 0.0156 & 0.0313 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0156 & 0.0156 & -0.0313 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0156 & -0.0156 & 0.0000 & 0.0313 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0156 & -0.0156 & 0.0000 & -0.0313 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

4. Main Result.

4.1.Homogeneous State Equations:

We would like to established a procedure to solve the homogeneous state equations via the rationalized Haar series, let rewrite $\dot{\boldsymbol{x}}(t)$ in terms of rationalized Haar functions as:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1\ (t) \\ \dot{\boldsymbol{x}}_2\ (t) \\ \dot{\boldsymbol{x}}_3\ (t) \\ \vdots \\ \dot{\boldsymbol{x}}_d\ (t) \end{bmatrix} = \begin{bmatrix} \alpha_{10}\ \alpha_{11}\ \alpha_{12}\ \cdots\ \alpha_{1\,(m-1)} \\ \alpha_{20}\ \alpha_{21}\ \alpha_{22}\ \cdots\ \alpha_{2\,(m-1)} \\ \alpha_{30}\ \alpha_{31}\ \alpha_{32}\ \cdots\ \alpha_{3\,(m-1)} \\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \\ \alpha_{d0}\ \alpha_{d1}\ \alpha_{d2}\ \cdots\ \alpha_{d\,(m-1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{RH}_0(t) \\ \boldsymbol{RH}_1(t) \\ \boldsymbol{RH}_2(t) \\ \vdots \\ \boldsymbol{RH}_{m-1}(t) \end{bmatrix} \quad \ldots (10)$$

Where, $\alpha_{ij}, i = 1,2,3,\ldots,n, j = 0,1,2,\ldots,m-1$ are constant to be determined. Eq. (10) can be also written by:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} \propto_1^T \\ \propto_2^T \\ \propto_3^T \\ \vdots \\ \propto_d^T \end{bmatrix} \boldsymbol{RH}_m(t) = \propto \boldsymbol{RH}_m\ (t) \ldots (11)$$

Now, integrate eq. (11) on $[0, t)$, we obtain:

$$\boldsymbol{x}(t) - \boldsymbol{x}(0) = \propto \int_0^t \boldsymbol{RH}_m\ (x)dx$$

Then, by using eq. (7), we get

$$\boldsymbol{x}(t) - \boldsymbol{x}(0) = \propto \boldsymbol{E}_m \boldsymbol{RH}_m\ (t)$$

$$\boldsymbol{x}(t) = \propto \boldsymbol{E}_m \boldsymbol{RH}_m(t) + \boldsymbol{x}(0) \quad \ldots (12)$$

Substituting eq. (11) and eq. (12) into eq. (2), yields

$$\propto \boldsymbol{RH}_m(t) = \boldsymbol{A}\ (\propto \boldsymbol{E}_m \boldsymbol{RH}_m(t) + \boldsymbol{x}(0))$$

$$\propto \boldsymbol{RH}_m(t) = \boldsymbol{A} \propto \boldsymbol{E}_m \boldsymbol{RH}_m(t) + \boldsymbol{Ax}(0)$$

Where, $\boldsymbol{Ax}(0)$ can be written as a form of vector, or

$$\boldsymbol{Ax}(0) = \boldsymbol{Ax}(0)\boldsymbol{RH}_0(t) = [\boldsymbol{Ax}(0)\quad \underbrace{0\ 0 \ldots 0]}_{(m-1)column}\ \boldsymbol{RH}_m(t) = \boldsymbol{GRH}_m(t)$$

Finally, we have:

$$\propto = \boldsymbol{A} \propto \boldsymbol{E}_m + \boldsymbol{G} \ldots (13)$$

And solving for $\propto$ we get

$$\begin{bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \\ \vdots \\ \propto_m \end{bmatrix} = [I - \boldsymbol{A} \otimes \boldsymbol{E}_m^T]^{-1} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_m \end{bmatrix} \ldots (14)$$

Where, G_1 is the first column of $\boldsymbol{G}$ and G_2 is the second column of $\boldsymbol{G}$, ..., etc., with I is the identity matrix and $\boldsymbol{A} \otimes \boldsymbol{E}_m^T$ is the Kronecker product defined as:

$$\boldsymbol{A} \otimes \boldsymbol{E}_m^T = \begin{bmatrix} e_{11}\boldsymbol{A}\ e_{21}\boldsymbol{A} \ldots e_{m1}\boldsymbol{A} \\ e_{12}\boldsymbol{A}\ e_{22}\boldsymbol{A} \ldots e_{m2}\boldsymbol{A} \\ \vdots\ \vdots\ \ \ldots\ \vdots \\ e_{1m}\boldsymbol{A}\ e_{2m}\boldsymbol{A} \ldots e_{mm}\boldsymbol{A} \end{bmatrix} \ldots (15)$$

After $\propto$ is determined the solution $\dot{\boldsymbol{x}}(t)$ is obtained. The solution $\boldsymbol{x}(t)$ is easily found by substituting $\propto$ into eq. (12).

Note: For more information and properties of Kronecker term with their applications, see[18].

Example (1): Consider the following homogeneous state equations (**Free system example**):

$$\dot{\boldsymbol{x}}(t) = -4\boldsymbol{x}(t) \quad \boldsymbol{x}(0) = 1$$

We should like to get the solution by the rationalized Haar series approach. Table (1) and table (2) shown the exact solution and approximation solutions in term of rationalized Haar series of order 4 and order 8.

Table (1): The comparison of exact solution and rationalized Haar series when $m = 4$.

Time (t)	Exact solution $x_1(t) = e^{-4t}$	Approximate solution $x_1(t)$	Error =|Exact- $x_1(t)$|
1/8	0.6065	0.6666	0.0601
3/8	0.2231	0.2222	0.0009
5/8	0.0821	0.0741	0.0080
7/8	0.0302	0.0247	0.0055

Table (2): The comparison of exact solution and rationalized Haar series when $m = 8$.

Time (t)	Exact solution $x_1(t) = e^{-4t}$	Approximate solution $x_1(t)$	Error =|Exact- $x_1(t)$|
1/16	0.7788	0.8000	0.0212
3/16	0.4724	0.4800	0.0076
5/6	0.2865	0.2880	0.0015
7/16	0.1738	0.1728	0.0010
9/16	0.1054	0.1036	0.0018
11/16	0.0639	0.0622	0.0017
13/16	0.0388	0.0374	0.0014
15/16	0.0235	0.0224	0.0011

We see that from table (1) and table (2) the rationalized Haar solution approximate to the exact solution when m is increasing, as well as error values are inversely proportional to m.

4.2Non-Homogeneous State Equations:

Let us consider the non homogeneous state equations which are described in eq. (1) by:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \;\; \boldsymbol{x}(0) = \boldsymbol{x}_0$$

Where, $\boldsymbol{x}\,(t)$ is a state vector of d-components and $\boldsymbol{u}(t)$ is an input vector of μ – components, $\boldsymbol{A}$ and $\boldsymbol{B}$ are $d \times d$ and $d \times \mu$ matrices respectively. We would like to establish a procedure to solve the non homogeneous state equations via the rationalized Haar series. For solving this problem, let the input vector $\boldsymbol{u}(t)$ can be expanded by the rationalized Haar series:

$$\boldsymbol{u}(t) = \begin{bmatrix} u_{10}\; u_{11}\; u_{12} & \cdots & u_{1\,(m-1)} \\ u_{20}\; u_{21}\; u_{22} & \cdots & u_{2\,(m-1)} \\ u_{30}\; u_{31}\; u_{32} & \cdots & u_{3\,(m-1)} \\ \vdots\;\;\vdots\;\;\vdots\;\;\vdots & & \vdots \\ u_{\mu 0}\; u_{\mu 1}\; u_{\mu 2} & \cdots & u_{\mu\,(m-1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{RH}_0(t) \\ \boldsymbol{RH}_1(t) \\ \boldsymbol{RH}_2(t) \\ \vdots \\ \boldsymbol{RH}_{m-1}(t) \end{bmatrix} = \boldsymbol{URH}_m(t) \; \ldots (16)$$

Where, $\boldsymbol{U}$ is a known $\mu \times m$ constant matrix. Substituting eq. (11), eq. (16) and eq. (12) into eq. (1), yields

$$\propto \boldsymbol{RH}_m(t) = \boldsymbol{A}\big(\propto \boldsymbol{E}_m\boldsymbol{RH}_m(t) + \boldsymbol{x}(0)\big)\boldsymbol{BURH}_m(t)$$

$$\propto \boldsymbol{RH}_m(t) = \boldsymbol{A} \propto \boldsymbol{E}_m\boldsymbol{RH}_m(t) + \boldsymbol{A}\boldsymbol{x}(0) + \; \boldsymbol{BURH}_m(t)$$

Finally, we get

$$\propto = \boldsymbol{A} \propto \boldsymbol{E}_m + \boldsymbol{G} + \boldsymbol{BU}$$

$$\propto = \boldsymbol{A} \propto \boldsymbol{E}_m + \boldsymbol{Q} \; \ldots (17)$$

Where, $\boldsymbol{Q} = \boldsymbol{G} + \boldsymbol{BU}$. By using the Kroncker product eq. (17) becomes:

$$\begin{bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \\ \vdots \\ \propto_m \end{bmatrix} = [I - \boldsymbol{A} \otimes \boldsymbol{E}_m^T]^{-1} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_m \end{bmatrix}$$

Where, q_1 is the first column of ; q_2 is the second column of $\boldsymbol{Q}$, ..., etc. After $\propto$ is determined the solution $\dot{\boldsymbol{x}}(t)$ is obtained. The solution $\boldsymbol{x}(t)$ is easily found by substituting $\propto$ into eq. (12)

$$\boldsymbol{x}(t) = \propto \boldsymbol{E}_m\boldsymbol{RH}_m(t) + \boldsymbol{x}(0) \; \ldots (18)$$

Eq. (18) is an approximation solution of eq. (1) via rationalized Haar functions.

Example (2): Consider the following non homogeneous state equations, [10]:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1\,(t) \\ \dot{\boldsymbol{x}}_2\,(t) \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1\,(t) \\ \boldsymbol{x}_2\,(t) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u\,(t)$$

With the initial conditions: $\begin{bmatrix} x_1\ (0) \\ x_2\ (0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $\boldsymbol{u}\ (t)$ is the unite-step input: $\boldsymbol{u(t)} = \begin{cases} \boldsymbol{0\ if\ t < 0} \\ \boldsymbol{1\ if\ t \geq 0} \end{cases}$

.The exact solution is

$$\begin{bmatrix} x_1\ (t) \\ x_2\ (t) \end{bmatrix} = \begin{bmatrix} e^{-0.5t}\sin\,(0.5t) \\ -e^{-0.5t}\ (\cos\,(0.5t + \sin(0.5t)) + 1 \end{bmatrix}$$

By using the above steps in solution of non-homogeneous via rationalized Haar functions, we have the following tables:

Table (3): The exact and approximation solution of non-homogeneous system example (2) with $\boldsymbol{m = 4}$.

Time	Exact Solution $x_1(t)$	Approximate solution $x_1(t)$	Error =|Exact-$x_1(t)$|
1/8	0.0587	0.0552	0.0035
3/8	0.1545	0.1518	0.0027
5/8	0.2249	0.2230	0.0019
7/8	0.2735	0.2723	0.0012

Time	Exact Solution $x_2(t)$	Approximate solution $x_2(t)$	Error =|Exact-$x_2(t)$|
1/8	0.0037	0.0069	0.0032
3/8	0.0310	0.0328	0.0018
5/8	0.0789	0.0796	0.0007
7/8	0.1416	0.1415	0.0001

Table (4): The exact and approximation solution of non-homogeneous system example (2) with $\boldsymbol{m = 8}$.

Time	Exact Solution $x_1(t)$	Approximate Solution $x_1(t)$	Error =|Exact-$x_1(t)$|	Exact Solution $x_2(t)$	Approximate Solution $x_2(t)$	Error =|Exact-$x_2(t)$|
1/16	**0.0303**	**0.0194**	**0.0009**	**0.0010**	**0.0018**	**0.0008**
3/16	**0.0852**	**0.0844**	**0.0008**	**0.0083**	**0.0090**	**0.0007**
5/16	**0.1331**	**0.1324**	**0.0007**	**0.0220**	**0.0224**	**0.0004**
7/16	**0.1744**	**0.1738**	**0.0006**	**0.0413**	**0.0416**	**0.0003**
9/16	**0.2095**	**0.2090**	**0.0005**	**0.0653**	**0.0655**	**0.0002**
11/16	**0.2390**	**0.2386**	**0.0004**	**0.0934**	**0.0935**	**0.0001**
13/16	**0.2632**	**0.2629**	**0.0003**	**0.1248**	**0.1249**	**0.0001**
15/16	**0.2827**	**0.2825**	**0.0002**	**0.1590**	**0.1589**	**0.0001**

5. Conclusion.

A rationalized Haar function method for solving the state equations of linear time invariant system is established. The basic formulas are eq. (12) and eq. (18) compared with the exact solutions in examples (1) and (2) respectively, the proposed approach is much simpler in analysis and easier in implementation.

6. Recommendations.

Through our paper, we recommend some recommendations:

1- Using of another complete orthonormal systems instead of rationalized Haar functions such as complete orthogonal polynomials approximation: Legendre polynomials, Laguerre polynomials, Tchebycheff polynomials of the first and second kind and others ….

2- Giving some practical examples in the control system and using such polynomials to get the suitable approximation for these examples, as well as, compared them with the using of the rationalized Harr functions.

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