Journal of Natural Sciences,
Life and Applied Sciences
Issue (2), Volume (3)
June 2019
ISSN: 2522-3356

المجلة العربية للعلوم و نشر الأبحاث
Arab Journal of Sciences & Research Publishing
AJSRP

Laplace–Elzaki Transform and its Properties with Applications to Integral and Partial Differential Equations

Safaa Adnan Shaikh Al-Sook **Mohammad Mahmud Amer**

Faculty of Science || Al-Baath University || Syria

Abstract: Laplace-Elzaki transform (LET) as a double integral transform of a function $f(x,t)$ of two variables was presented to solve some integral and partial differential equations. Main properties and theorems were proved. The convolution of two function $f(x,t)$ and $g(x,t)$ and the convolution theorem were discussed. The integral and partial differential equations were turned to algebraic ones by using (LET) and its properties. The results showed that the Laplace-Elzaki transform was more efficient and useful to handle such these kinds of equations.

Keywords: Laplace–Elzaki transform, Laplace transform, Elzaki transform, Convolution, Integral and partial differential equations.

تحويل لابلاس-الزاكي وخواصه مع تطبيقاته على المعادلات التكاملية والتفاضلية الجزئية

صفاء عدنان شيخ السوق **محمد محمود عامر**

كلية العلوم || جامعة البعث || سوريا

الملخص: في هذه المقالة تمت دراسة تحويل لابلاس- الزاكي كتحويل تكاملي مضاعف لحل بعض المعادلات التكاملية والتفاضلية الجزئية كذلك تم إثبات خواص ومبرهنات أساسية له بالإضافة إلى تلاف دالتين $f(x,t)$ و $g(x,t)$ من أجل التطرق لمبرهنة خاصة بهذا التلاف. هذا وعند تطبيق تحويل لابلاس- الزاكي وخواصه على معادلات تكاملية وتفاضلية جزئية تحوّلت هذه المعادلات إلى معادلات جبرية وأشارت النتائج إلى أنّ هذا التحويل أكثر فعّالية وفائدة لحل هذا النوع من المعادلات.

الكلمات المفتاحية: تحويل لابلاس-الزاكي، تحويل لابلاس، تحويل الزاكي، التلاف (الطي)، معادلات تكاملية وتفاضلية جزئية.

1. Introduction :

Pierre – Simon Laplace (1782) introduced the idea of its transform that became one of the most famous transforms in mathematics, physics and engineering sciences. The Laplace transform was used to find out the solution of linear differential, difference, and integral equations [1,2,3]. On ther hand, Tarig.M.Elzaki (2010) modified Sumudu transform [4] and gave a new technique that used to solve linear and nonlinear differential equations [5-8], integral equations[9], and other applications .Whereas using single Laplace and Elzaki transforms to solve equations with unkown function of two variables were hard and useless sometimes so the matheticians as L.Debnath used the double Laplace transform to solve functional, double integral equations and partial differential equations [10]. While T.M.Elzaki and E.M.A.Hilal solved telegraph partial differential equation by using double Elzaki transform [11]. In this paper , another double transform which called Laplace-Elzaki transform of function $f(x,t)$ was studied with properties and theorems to find out the solution of some integral and partial differential equations. Consequently, the aim of this work is to develop a method to solve double integral and partial differential equations easily by turning these kind of equations to algebraic ones.

2. Research problem:

Partial differential equations and double integral equations with convolution type are used to describe many problems in the field of engineering and most applied science . The solving of these equations by Using single transforms were more difficult than using the double transforms. The Laplace-Elzaki transform as a double transform and its properties were discussed to solve these kind of equations easily.

3. Materials and methods of research:

The inductive thinking was used to get a double integral transform. The single Laplace and Elzaki transforms were combined in a double integral transform which called Laplace-Elzaki transform , so most of properties for the two single transforms were generalized and most results related to the gamma function.

3.1. Basic concepts:

3.1.1. Laplace transform [2]:

Let $f(x)$ is a piecewise continuous function on interval $[0,\infty[$ and of exponential order so it satisfy :

$$|f(x)| < Me^{ax} \ ; \ a > 0$$

Then the Laplace transform of this function is defined by :

$$L(s) = L\big(f(x)\big) = \int_0^\infty e^{-sx} f(x)dx \quad (1)$$

The inverse Laplace transform is :

$$f(x) = L^{-1}\big(L(s)\big) = \int_0^\infty e^{sx} L(s)ds \quad (2)$$

3.1.2. Elzaki transform [5-7]:

If $f(t)$ is a function from the set A where:

$$A=\left\{f(t):\exists M,k_1,k_2>0,|f(t)|<Me^{\frac{|t|}{k_j}};t\in(-1)^j\times[0,\infty[\right\}$$

Then the Elzaki transform of this function is:

$$E(p)=E\big(f(t)\big)=p\int_0^\infty e^{-\frac{t}{p}}f(t)dt \quad (3)$$

And the relation between Laplace and Elzaki transform is :

$$L(s)=pE\left(\frac{1}{p}\right) \qquad (4)$$

So the inverse Elzaki transform has the form :

$$f(t)=E^{-1}\big(E(p)\big)=\int_0^\infty e^{st}\,pE\left(\frac{1}{p}\right)dp \quad (5)$$

3.1.3. Gamma function[12]:

$\Gamma(a)$ is the gamma function which defined by the form

$$\Gamma(a)=\int_0^\infty e^{-x}x^{a-1}dx \qquad (6)$$

The gamma function satisfies many properties like :

$$\Gamma(a+1)=a\Gamma(a) \qquad (7)$$

If $a\in N$ then,

$$\Gamma(\mathrm{a})=(\mathrm{a}+1)! \qquad (8)$$

4. Discussion and results:

4.1. Definition of the Laplace-Elzaki transform (LET) :

The Laplace-Elzaki transform of function $f(x,t)$ of two variable x and t defined in the first quadrant of the $x-t$ plane is defined by the double integral in the form:

$$\bar{\bar{f}}(s,p)=L_xE_t\big(f(x,t)\big)=LE\big(f(x,t)\big) \qquad =p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}f(x,t)dxdt \quad (9)$$

Evidently, LET is linear integral transformation as shown below :

$$LE\big(\alpha f(x,t)+\beta g(x,t)\big)=p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\,[\alpha f(x,t)+\beta g(x,t)]dxdt$$

$$=p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\alpha f(x,t)dxdt+p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\beta g(x,t)dxdt$$

$$=\alpha p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}f(x,t)dxdt+\beta p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}g(x,t)dxdt$$

$$=\alpha LE\big(f(x,t)\big)+\beta LE\big(g(x,t)\big) \qquad (10)$$

Where α and β are constants.

By using the Bromwich inversion formula [2] The inverse Laplace-Elzaki transform is defined by the complex integral formula :

$$f(x,t)=LE^{-1}\left(\bar{\bar{f}}\,(s,p)\right)$$

$$=\frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} pe^{sx+\frac{t}{p}}f\left(s,\frac{1}{p}\right)dsdp \ (11)$$

4.2. Laplace – Elzaki transform of basic functions :

(a)

$f(x,t)=x^a t^b$ for $x>0$ and $t>0$, then

$$\bar{\bar{f}}\,(s,p)=LE(x^a t^b)=p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}x^a t^b dxdt$$

$$=\int_0^\infty e^{-sx}x^a dx\int_0^\infty pe^{-\frac{t}{p}}t^b dt$$

$$=\frac{\Gamma(a+1)}{s^{a+1}}p^{b+2}\Gamma(b+1) \qquad (12)$$

In particular,

$$LE(1)=\frac{p^2}{s} \qquad (13)$$

where $a>-1$ and $b>-1$ are real numbers

Consequently if a and b are natural numbers in (12) we obtain

$$LE(x^a t^b)=\frac{a!\,b!}{s^{a+1}}p^{b+2} \qquad (14)$$

(b)

Let $f(x,t)=e^{(ax+bt)}$, then

$$\bar{\bar{f}}\,(s,p)=LE\big(e^{(ax+bt)}\big)=p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}e^{(ax+bt)}dxdt$$

$$= \int_0^\infty e^{-(s-a)x}\, dx \int_0^\infty p e^{-(\frac{1}{p}-b)t}\, dt$$

$$= \frac{p^2}{(s-a)(1-bp)} \;;\; Re(\tfrac{1}{p}) > b \;\&\; Re(s) > a \quad (15)$$

Similarly,

$$LE\left(e^{i(ax+bt)}\right) = \frac{p^2}{(s-ia)(1-ibp)} \quad (16)$$

(*c*)

As we known

$$\sin(ax+bt) = \frac{e^{i(ax+bt)} - e^{-i(ax+bt)}}{2i} \quad (17)$$

$$\cos(ax+bt) = \frac{e^{i(ax+bt)} + e^{-i(ax+bt)}}{2} \quad (18)$$

From the linearity property of (LET) and the equations (15), (16), (17), and (18) we get :

$$\mathrm{LE}[\sin(ax+bt)] = \frac{ap^2 + sbp^3}{(s^2+a^2)(1+b^2p^2)} \quad (19)$$

$$\mathrm{LE}[\cos(ax+bt)] = \frac{sp^2 - abp^3}{(s^2+a^2)(1+b^2p^2)} \quad (20)$$

Similarly,

$$\sinh(ax+bt) = \frac{e^{(ax+bt)} - e^{-(ax+bt)}}{2} \quad (21)$$

$$\cosh(ax+bt) = \frac{e^{(ax+bt)} + e^{-(ax+bt)}}{2} \quad (22)$$

So,

$$\mathrm{LE}[\sinh(ax+bt)] = \frac{ap^2 + sbp^3}{(s^2-a^2)(1-b^2p^2)} \quad (23)$$

$$\mathrm{LE}[\cosh(ax+bt)] = \frac{sp^2 - abp^3}{(s^2-a^2)(1-b^2p^2)} \quad (24)$$

(d)

$$LE\left(J_0(a\sqrt{xt})\right) = p\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{p}} J_0(a\sqrt{xt})\,dx\,dt$$

$$= p\int_0^\infty e^{-\frac{t}{p}}\, dt \int_0^\infty e^{-sx} J_0(a\sqrt{xt})\,dx$$

$$= \frac{p}{s}\int_0^\infty e^{-\frac{t}{p}} e^{-\frac{a^2t}{4s}}\, dt = \frac{1}{s} E\left(e^{-\frac{a^2t}{4s}}\right)$$

$$= \frac{p^2}{s(1+\frac{a^2}{4s}p)} = \frac{4p^2}{4s+a^2p} \quad (25)$$

Where $J_0(x)$ is a Bessel function of zero order which has the formula:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n} \quad (26)$$

(e)

$$LE[H(x-x_0, t-t_0)] = p\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{p}} H(x-x_0, t-t_0)dxdt$$

$$= \int_{x_0}^{\infty} e^{-sx}\,dx \int_{t_0}^{\infty} pe^{-\frac{t}{p}}\,dt$$

$$= p^2 e^{-\frac{t_0}{p}} \frac{e^{-sx_0}}{s} \quad (27)$$

where $H(x-x_0, t-t_0)$ is Heaviside function which has the formula :

$$H(x-x_0, t-t_0) = \begin{cases} 1\,; & x \geq x_0 \text{ and } t \geq t_0 \\ 0\,; & x < x_0 \text{ and } t < t_0 \end{cases} \quad (28)$$

A table of Laplace-Elzaki transform can be constructed from the standard tables of Laplace and Elzaki transforms by using the definition (9) or by evaluating double integrals. The above results can be used to solve integral and partial differential equations.

4.2.1. Corollary: Let $f(x,t) = g(x)h(t)$, then

$$\bar{\bar{f}}\,(s,p) = LE[f(x,t)] = L[g(x)]E[h(t)] \quad (29)$$

Proof:

$$\bar{\bar{f}}\,(s,p) = LE[f(x,t)] = p\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{p}} g(x)h(t)dxdt$$

$$= \int_0^\infty e^{-sx} g(x)dx \int_0^\infty pe^{-\frac{t}{p}} h(t)dt = L[g(x)]E[h(t)]$$

4.2.2. Corollary: If $\bar{\bar{f}}\,(s,p) = L[g(x)]E[h(t)] = L(s)E(p)$ then the inverse Laplace-Elzaki transform can be calculated by the residue theorem [2] :

$$f(x,t) = LE^{-1}(\bar{\bar{f}}\,(s,p))$$

$$= \frac{1}{(2\pi i)^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\beta-i\infty}^{\beta+i\infty} pe^{sx+pt} L(s)E\left(\frac{1}{p}\right) dsdp$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} Res_{s=s_i}[e^{sx}L(s)]\, Res_{p=p_j}\left[pe^{pt}E(\tfrac{1}{p})\right] \quad (30)$$

Where $s_1, \dots\dots, s_n$ and $p_1, \dots\dots, p_m$ are the poles of $\bar{\bar{f}}\,(s,p) = L(s)E(p)$.

4.3. Existence conditions for Laplace-Elzaki transform :

If $f(x,t)$ is a function of exponential order then for $a > 0, b > 0$, $0 \le x < \infty$ and $0 \le t < \infty$ there exists a positive constant k such that for all $x > X$ and $t > \mathrm{T}$:

$$|f(x,t)| \le k e^{ax+bt} \qquad (31)$$

and we write

$$\mathrm{f(x,t)} = O(e^{ax+bt}) \text{ as } x \to \infty\,, t \to \infty \qquad (32)$$

Or, equivalently,

$$\lim_{x\to\infty, t\to\infty} e^{-\alpha x-\beta t}\,|f(x,t)| == k \lim_{x\to\infty, t\to\infty} e^{-(\alpha-a)x-(\beta-b)t} = 0;\ \alpha > a\,, \beta > b \ (33)$$

consequantly, if $f(x,t)$ is a continuous function in every finite intervals $(0,X)$ and $(0,T)$ also for $b = \frac{1}{c}\ (c > 0)$ a function $f(x,t)$ is of exponential order $e^{ax+\frac{t}{c}}$:

$$|f(x,t)| \le k e^{ax+\frac{t}{c}} \qquad (34)$$

then the Laplace-Elzaki transform of $f(x,t)$ exists for all s and p provided $Re(s) > a$ and $Re\left(\frac{1}{p}\right) > \frac{1}{c}$.

4.4. Basic properties of the Laplace-Elzaki transform:

We can prove the following general properties of the Laplace-Elzaki transform under suitable conditions of $f(x,t)$:

4.4.1. Shifting property :

$$LE\left(e^{(ax+bt)}f(x,t)\right) = (1-bp)LE\left(s-a, \frac{p}{1-bp}\right) \ (35)$$

Proof:

$$LE\left(e^{(ax+bt)}f(x,t)\right) = p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\,e^{(ax+bt)}f(x,t)dxdt$$

$$= p\int_0^\infty\int_0^\infty e^{-(\frac{1}{p}-b)t}\,e^{-(s-a)x}f(x,t)dxdt$$

Let $q = \frac{p}{1-bp}$ then,

$$LE\left(e^{(ax+bt)}f(x,t)\right) = (1-bp)q\int_0^\infty\int_0^\infty e^{-\frac{t}{q}}\,e^{-(s-a)x}f(x,t)dxdt$$

$$= (1-bp)LE(s-a,q) = (1-bp)LE\left(s-a, \frac{p}{1-bp}\right)$$

4.4.2. Derivative property :

If $\bar{\bar{f}}\,(s,p) = LE\big(f(x,t)\big)$ then,

a) $LE\left(\frac{\partial f(x,t)}{\partial x}\right) = s\bar{\bar{f}}\,(s,p) - E\big(f(0,t)\big)$ (36)

Proof:

$$LE\left(\frac{\partial f(x,t)}{\partial x}\right) = p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\,\frac{\partial f(x,t)}{\partial x}\,dxdt$$

$$= p\int_0^\infty e^{-\frac{t}{p}}\,dt\int_0^\infty e^{-sx}\,\frac{\partial f(x,t)}{\partial x}\,dx$$

Suppose $u = e^{-sx}$, $dv = \frac{\partial f(x,t)}{\partial x}dx$ then,

$$LE\left(\frac{\partial f(x,t)}{\partial x}\right) = p\int_0^\infty e^{-\frac{t}{p}}\,dt\left[e^{-sx}f(x,t)\Big|_{x=0}^{\infty} + s\int_0^\infty e^{-sx}\,f(x,t)dx\right]$$

$$= -E\big(f(0,t)\big) + s\bar{\bar{f}}\,(s,p)$$

b) $LE\left(\frac{\partial f(x,t)}{\partial t}\right) = \frac{1}{p}\bar{\bar{f}}\,(s,p) - pL\big(f(x,0)\big)$ (37)

proof:

$$LE\left(\frac{\partial f(x,t)}{\partial t}\right) = p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\,\frac{\partial f(x,t)}{\partial t}\,dxdt$$

$$= p\int_0^\infty e^{-sx}\,dx\int_0^\infty e^{-\frac{t}{p}}\,\frac{\partial f(x,t)}{\partial t}\,dt$$

Suppose $u = e^{-\frac{t}{p}}$, $dv = \frac{\partial f(x,t)}{\partial t}dt$ then,

$$LE\left(\frac{\partial f(x,t)}{\partial t}\right) = p\int_0^\infty e^{-sx}\,dx\left[e^{-\frac{t}{p}}f(x,t)\Big|_{t=0}^{\infty} + \frac{1}{p}\int_0^\infty e^{-\frac{t}{p}}\,f(x,t)dt\right]$$

$$= -pL\big(f(x,0)\big) + \frac{1}{p}\bar{\bar{f}}\,(s,p)$$

Similarly,

c) $LE\left(\frac{\partial^2 f(x,t)}{\partial x\partial t}\right) = \frac{s}{p}\bar{\bar{f}}\,(s,p) - spL\big(f(x,0)\big) - E\big(f_t(0,t)\big)$ (38)

d) $LE\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right) = s^2\bar{\bar{f}}\,(s,p) - sE\big(f(0,t)\big) - E\big(f_x(0,t)\big)$ (39)

e) $LE\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = \frac{1}{p^2}\bar{\bar{f}}\,(s,p) - L\big(f(x,0)\big) - pL\big(f_t(x,0)\big)$ (40)

4.4.3. Theorem : If $f(x,t)$ is periodic function of periods a and b (that is $f(x+a,t+b)=f(x,t)$ for all x and t) and if $LE(f(x,t))$ exists, then

$$LE\big(f(x,t)\big)=p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}f(x,t)dxdt$$

$$=p\int_0^a\int_0^b e^{-sx-\frac{t}{p}}f(x,t)dxdt+p\int_a^\infty\int_b^\infty e^{-sx-\frac{t}{p}}f(x,t)dxdt$$

Setting $u=x-a$, $v=t-b$, then :

$$LE\big(f(x,t)\big)=\bar{\bar{f}}\,(s,p)$$

$$=\ p\int_0^a\int_0^b e^{-sx-\frac{t}{p}}f(x,t)dxdt$$

$$+pe^{-sa-\frac{b}{p}}\int_0^\infty\int_0^\infty e^{-su-\frac{v}{p}}f(u+a,v+b)dudv$$

$$\bar{\bar{f}}\,(s,p)=\ p\int_0^a\int_0^b e^{-sx-\frac{t}{p}}f(x,t)dxdt+pe^{-sa-\frac{b}{p}}\int_0^\infty\int_0^\infty e^{-su-\frac{v}{p}}f(u,v)dudv$$

$$\bar{\bar{f}}\,(s,p)=p\int_0^a\int_0^b e^{-sx-\frac{t}{p}}f(x,t)dxdt+e^{-sa-\frac{b}{p}}\bar{\bar{f}}\,(s,p)$$

Consequently,

$$\bar{\bar{f}}\,(s,p)=\left[1-e^{-sa-\frac{b}{p}}\right]^{-1}.p\int_0^a\int_0^b e^{-sx-\frac{t}{p}}f(x,t)dxdt \quad (41)$$

4.5. Convolution and convolution theorem of the Laplace-Elzaki transform:

The convolution of $f(x,t)$ and $g(x,t)$ is denoted by $(f**g)(x,t)$ and defined by :

$$(f**g)(x,t)=\int_0^x\int_0^t f(x-u,t-v)\,g(u,v)dudv \quad (42)$$

The convolution is commutative, that is:

$$(f**g)(x,t)=(g**f)(x,t) \quad (43)$$

This follows from the definition (42), it can easily be verified that the following properties of convolution are correct :

1. $[f**(g**h)](x,t)=[(f**g)**h](x,t)$ (44) (Associative)
2. $[f**(ag+bh)](x,t)=a(f**g)(x,t)+b(f**h)(x,t)$ (45) (Distributive).
3. $(f**\delta)(x,t)=(\delta**f)(x,t)=f(x,t)$ (46) (Identity).

Where $\delta(x,t)$ is the Dirac delta function of x and t.

4.5.1. Theorem: If $\bar{\bar{f}}\ (s,p) = LE\big(f(x,t)\big)$, then

$$LE(f(x-\varepsilon,t-\tau)H(x-\varepsilon,t-\tau)) = e^{-s\varepsilon-\frac{\tau}{p}}\bar{\bar{f}}\ (s,p) \quad (47)$$

Where $H(x,t)$ is the Heaviside unit step function defined by (28).

Proof: We have, by definition:

$$LE(f(x-\varepsilon,t-\tau)H(x-\varepsilon,t-\tau)) =$$

$$= p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}f(x-\varepsilon,t-\tau)H(x-\varepsilon,t-\tau)\ dxdt$$

$$= p\int_\varepsilon^\infty\int_\tau^\infty e^{-sx-\frac{t}{p}}\ f(x-\varepsilon,t-\tau)dxdt$$

by putting $x-\varepsilon = y\ , t-\tau = z$:

$$= pe^{-s\varepsilon-\frac{\tau}{p}}\int_0^\infty\int_0^\infty e^{-sy-\frac{z}{p}}\ f(y,z)dydz$$

$$= e^{-s\varepsilon-\frac{\tau}{p}}\bar{\bar{f}}\ (s,p)$$

4.5.2. Theorem: (Convolution theorem)

If $\bar{\bar{f}}\ (s,p) = LE\big(f(x,t)\big)$ and $\bar{\bar{g}}\ (s,p) = LE\big(g(x,t)\big)$, then

$$LE[(f**g)(x,t)] = \frac{1}{p}LE\big(f(x,t)\big)LE\big(g(x,t)\big) = \frac{1}{p}\bar{\bar{f}}\ (s,p)\bar{\bar{g}}\ (s,p) \quad (48)$$

Or, equivalently,

$$LE^{-1}\left[\frac{1}{p}\ \bar{\bar{f}}\ (s,p).\bar{\bar{g}}\ (s,p)\right] = (f**g)(x,t) \quad (49)$$

Where $(f**g)(x,t)$ is defined by the double integral (42) of $f(x,t)$ and $g(x,t)$.

Proof: We have, by definition,

$$LE[(f**g)(x,t)] = p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}(f**g)(x,t)dxdt$$

$$= p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\left[\int_0^x\int_0^t f(x-u,t-v)\ g(u,v)dudv\right]dxdt$$

Which is using the Heaviside unit step function,

$$= p\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}\left[\int_0^\infty\int_0^\infty f(x-u,t-v)H(x-u,t-v)\ g(u,v)dudv\right]dxdt$$

$$= p\int_0^\infty\int_0^\infty g(u,v)dudv\left[\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{p}}f(x-u,t-v)H(x-u,t-v)dxdt\right]$$

Which is, by theorem (4.5.1),

$$=\int_0^\infty\int_0^\infty g(u,v)e^{-su-\frac{v}{p}}\bar{\bar{f}}\,(s,p)dudv=\frac{1}{p}\,.\bar{\bar{f}}\,(s,p).p\int_0^\infty\int_0^\infty e^{-su-\frac{v}{p}}g(u,v)dudv$$

$$=\frac{1}{p}\,\bar{\bar{f}}\,(s,p)\bar{\bar{g}}\,(s,p)$$

4.6. Applications of Laplace-Elzaki transform to integral and partial Differential equations:

4.6.1. Integral equations:

(a) The nonhomogeneous two dimensional Voltera integral equation of the first kind of convolution type is defined as:

$$\lambda\int_0^x\int_0^t k(x-u,t-v)\,f(u,v)dudv=g(x,t)\quad(50)$$

Where $f(x,t)$ is an unknown function and λ is constant.

We apply the Laplace-Elzaki transform to (50) to obtain:

$$LE\left[\lambda\int_0^x\int_0^t k(x-u,t-v)\,f(u,v)dudv\right]=LE[g(x,t)]$$

Using of (42) and (48) to get:

$$\lambda LE[(k**f)(x,t)]=LE[g(x,t)]$$

$$\frac{\lambda}{p}\,\bar{\bar{k}}\,(s,p).\bar{\bar{f}}\,(s,p)=\ \bar{\bar{g}}\,(s,p)$$

Simplifying this equation :

$$\bar{\bar{f}}\,(s,p)=\frac{p}{\lambda}\frac{\bar{\bar{g}}\,(s,p)}{\bar{\bar{k}}\,(s,p)}=\frac{1}{\lambda}H(s,p)$$

Where $H(s,p)=p\,\frac{\bar{\bar{g}}\,(s,p)}{\bar{\bar{k}}\,(s,p)}$

The inverse transform gives the solution of the equation (50) as:

$$f(x,t)=\frac{1}{\lambda}LE^{-1}[H(s,p)]$$

Notice that according to the convolution property (43) the equation (50) can be defined by :

$$\lambda\int_0^x\int_0^t k(u,v)\,f(x-u,t-v)dudv=g(x,t)\quad(51)$$

Examples:

(a) Solve the equation:

$$2\int_0^x\int_0^t e^{u-v} f(x-u,t-v)dudv = xe^{x-t} - xe^x \quad (52)$$

Application of the Laplace-Elzaki transform to (52) next Using (42) and (48) gives :

$$2LE(f(x,t) ** e^{x-t}) = LE(xe^{x-t}) - LE(xe^x)$$

Or,

$$\frac{2}{p}\bar{\bar{f}}\,(s,p)\frac{p^2}{(s-1)(1+p)} = \frac{p^2}{(1+p)(s-1)^2} - \frac{p^2}{(s-1)^2}$$

$$\bar{\bar{f}}\,(s,p) = -\frac{1}{2}\frac{p^2}{s-1}$$

If the inverse transform was applied on the last relation we obtain :

$$f(x,t) = -\frac{1}{2}e^x$$

Which is the solution of (52).

(b) Solve the equation :

$$\int_0^x\int_0^t f(x-u,t-v)\,f(u,v)dudv = a^2t\,;a = \text{const} \quad (53)$$

If we apply the Laplace-Elzaki transform on (53) and depend on (42) and (48) we obtain:

$$LE(f ** f) = LE(a^2t)$$

Or,

$$\frac{1}{p}\bar{\bar{f}}\,(s,p)\bar{\bar{f}}\,(s,p) = a^2\frac{p^3}{s}$$

$$\left[\bar{\bar{f}}\,(s,p)\right]^2 = a^2\frac{p^4}{s}$$

So,

$$\bar{\bar{f}}\,(s,p) = \frac{ap^2}{\sqrt{s}}$$

If we take the inverse transform then:

$$f(x,t) = \frac{a}{\sqrt{\pi}}\frac{1}{\sqrt{x}}$$

That is the solution of (53).

4.6.2. Partial differential equations (PDE):

The nonhomogeneous PDE of second order and constant coefficients which have the form:

$$au_{xx} + bu_{tt} + cu_x + du_t + eu(x,t) = f(x,t) \quad (54)$$

Where a, b, c, d, e are constants, with initial and boundary conditions:

$$u(x,0) = f_1(x) \ , u_t(x,0) = f_2(x) \quad (55)$$
$$u(0,t) = f_3(t) \ , u_x(0,t) = f_4(t) \quad (56)$$

Apply Laplace-Elzaki transform ($\bar{\bar{u}}\,(s,p) = LE[u(x,t)]$) with linearity and derivatives properties to (54) to get:

$$aLE(u_{xx}) + bLE(u_{tt}) + cLE(u_x) + dLE(u_t) + eLE(u) = LE(f(x,t))$$

Or,

$$a\left[s^2\bar{\bar{u}}\,(s,p) - E\big(u(0,t)\big) - E\big(u_x(0,t)\big)\right] + b\left[\frac{1}{p^2}\bar{\bar{u}}\,(s,p) - L\big(u(x,0)\big) - pL\big(u_t(x,0)\big)\right] + c\left[s\bar{\bar{u}}\,(s,p) - E\big(u(0,t)\big)\right] + d\left[\frac{1}{p}\bar{\bar{u}}\,(s,p) - pL\big(u(x,0)\big)\right] = \bar{\bar{f}}\,(s,p)$$

By using (55)and (56) we obtain :

$$\left(as^2 + \frac{b}{p^2} + cs + \frac{d}{p} + e\right)\bar{\bar{u}}\,(s,p) = as\bar{f_3}(p) + a\bar{f_4}(p) + b\bar{f_1}(s) + +bp\bar{f_2}(s) + c\bar{f_3}(p) + ep\bar{f_1}(s) + \bar{\bar{f}}\,(s,p)$$

Where $\bar{f_1}(s) = L\big(u(x,0)\big), \bar{f_2}(s) = L\big(u_t(x,0)\big), \bar{f_3}(p) = E\big(u(0,t)\big)$, $\bar{f_4}(p) = E(u_x(0,t))$.

Simplify the last relation next apply the inverse Laplace-Elzaki transform to obtain the solution $u(x,t)$ of the given equation.

Example(1): solve the equation

$$u_{xx} - u_{tt} + u_t + 9u = sin3x \quad (57)$$

With

$$u(x,0) = 0 \ , u_t(x,0) = sin3x \quad (58)$$
$$u(0,t) = 0 \ , u_x(0,t) = 3t \quad (59)$$

Application of the Laplace-Elzaki transform to (57), Laplace transform to (58) and Elzaki transform to (59) give:

$$LE(u_{xx}) - LE(u_{tt}) + LE(u_t) + 9LE(u) = LE(sin3x)$$

Let $LE\big(u(x,t)\big) = \bar{\bar{u}}\,(s,p)$ and use (37),(39) and (40) then:

$$s^2\bar{\bar{u}}\,(s,p) - 3p^3 - \frac{1}{p^2}\bar{\bar{u}}\,(s,p) + \frac{3p}{s^2+9} + \frac{1}{p}\bar{\bar{u}}\,(s,p) + 9\bar{\bar{u}}\,(s,p) = \frac{3p^2}{s^2+9}$$

Or,

$$\bar{\bar{u}}\,(s,p) = \frac{3p^3}{s^2+9}$$

The inverse Laplace-Elzaki transform gives :

$$u(x,t) = tsin3x$$

Remark: If $f(x,t) = 0$ in equation (54) then we called this PDE a homogeneous equation.

Example(2):Solve the equation

$$u_{xx} + 2u_{tt} + 3u_x = 0 \quad (60)$$

With

$$u(x,0) = 0 \;, u_t(x,0) = e^{-3x} \;(61)$$
$$u(0,t) = t \;, u_x(0,t) = -3t \;(62)$$

Apply the Laplace-Elzaki transform on (60), Laplace transform on (61) and Elzaki transform on (62), next use (36), (39) and (40) to obtain:

$$LE(u_{xx}) + 2LE(u_{tt}) + 3LE(u_x) = 0$$

Let $LE(u(x,t) = \bar{\bar{u}}\,(s,p)$ then,

$$s^2\bar{\bar{u}}\,(s,p) - sp^3 + 3p^3 + \frac{2}{p^2}\bar{\bar{u}}\,(s,p) - \frac{2p}{s+3} - 3p^3 = 0$$

Or,

$$\left(s^2 + \frac{2}{p^2} + 3s\right)\bar{\bar{u}}\,(s,p) = sp^3 - 3p^3 + \frac{2p}{s+3} + 3p^3$$

Or,

$$\bar{\bar{u}}\,(s,p) = \frac{p^3}{s+3}$$

Applying inverse transform gives the solution :

$$u(x,t) = te^{-3x}$$

Conclusion:

Two different single transforms were combined together to present a new double transform (LET), so main properties and theorems were generalized with proofs to solve some integral and patial differential equations. We concluded that (LET) was a powerful technique to deal with these kind of equations. Another applications on integro-partial differential and nonlinear partial differential equations will be discussed in subsequent paper.

Reference:

1. J.L.Schiff, The Laplace Trasform Theory and Applications, Springer, New York, 1972.
2. L.Debnath, D.Bhatta, Integral Transforms and Their Applications Third Edition, CRC press, Chapman& Hall, Boca Raton, 2015.
3. R.Churchill, Operational Mathematics Third Edition, McGraw- Hill, New York, 1972.
4. F.B.M.Belgacem, A.A.Karaballi, "Sumudu Transform Fundamental Properties Investigations and Applications", Journal of Applied Mathematics and Stochastic Analysis, 1-23, 2006.
5. T.M.Elzaki, S.M.Elzaki, E.A.Elnour, "On the New Integral Transform "Elzaki transform" fundamental properties investigations and applications", Global Journal of Mathematical sciences; theory and practical,4, 1, 1-13, 2012.
6. T.M.Elzaki, S.M.Elzaki, E.M.A.Hilal, "Elzaki and Sumudu Transforms for Solving Differential Equations," Global Journal of Pure and Applied Mathematics, 8, 167-173, 2012.
7. T.M.Elzaki, S.A.Alkateeb, "Modification of Sumudu Transform "Elzaki transform" and Domain Decomposition Method", Applied Mathematical sciences, 9,13,603-611, 2015.
8. T.M.Elzaki, E.M.A.Hilal, "Solution of Nonlinear Differential equations Using Mixture of Elzaki Transform and Differential Transform Method," International Mathematica Forum, 7, 631-638, 2012.
9. Y.Song, H.Kim, "The Solution of Volterra Integral equation of The Second Kind by Using The Elzaki Transform," Applied Mathematical Science, 8, 525-530, 2014.
10. L.Debnath, "The Double Laplace Transforms and Their Applications to Functional, Integral and Partial Differential equations," Int. J. Appl. Comput. Math, 2,223-241, 2016.
11. T.M.Elzaki, E.M.A.Hilal, "Solution of telegraph Equation by Modified of Double Sumudu Transform "Elzaki Transform" ", Mathematical Theory and Modeling, 2, 4, 95-103, 2012.
12. W.Freeden and M.Gutting, Special Functions of Mathmatical (Geo-) Physics, Birkhauser, Basel, 2013.