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State Parameterization Basic Spline Functions for Trajectory Optimization

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Abstract: An important type of basic functions named basis spline (B-spline) is provided a simpler approximate and more stable approach to solve problems in optimal control. Furthermore, it can be proved that with special knot sequence, the B-spline basis are exactly Bernstein polynomials. The approximate technique is based on state variable is approximate as a linear combination of B-spline then anon linear optimization problem is obtained and the optimal coefficients are calculated using an iterative algorithm. Two different examples are tested using the proposed algorithm.

Keywords: Basic spline, Bernstein polynomials, state parameterization algorithm, optimal control problems (OCPs).

دوال سبلاين الاساسية لمعلمات الحالة لأمثلية المسار

مها مصدق دلفي
سهى نجيب شهاب
قسم العلوم التطبيقية || الجامعة التكنولوجية || بغداد || العراق

الملخص: تم اقتراح تقريب مبسط وأسلوب مستقر لحل مسائل في السيطرة المثلى بالاعتماد على نوع من الدوال الأساسية المهمة والمسماة دوال سبلاين الأساسية. إضافة إلى ذلك، تم برهنة أن دوال السبلاين الأساسية هي بالضبط متعددات حدود بيرنشتاين مع متسلسلة نقاط خاصة. الأسلوب التقريبي يعتمد على متغير الحالة حيث تم تقريبه كتقريب خطي من دوال السبلاين الأساسية وتم تحويل المسألة إلى مسألة غير خطية وتم حساب المعاملات باستخدام خوارزمية تكرارية. تم تطبيق مثالين للخوارزمية المقترحة.

الكلمات المفتاحية: سبلاين الأساسية، متعددات حدود برنشتاين، خوارزمية معلمة الحالة، مسائل السيطرة المثلى.

1- Introduction

The optimal control problem (OCP) is to obtain a control function that minimizes or maximizes known performance index governing by the system state equations together with the constraints. Their applications appear in many disciplines, economics, management, and engineering [1-3].

The basic OCP consists of three elements, first, the mathematical model of the controlled system that is either differential equation, integral equation of partial differential equation. Second, A set of boundary conditions concerning the value of the state system at initial time. Third, the performance index, which is minimized or maximized, is expressed mathematically in form of a scalar function.

The following optimal control problems are considered in this work. The optimum performance index is

$$J = \int_0^1 F(\tau, x(\tau), u(\tau) d\tau \quad (1)$$

Subject to the process illustrated by the differential equation on the time interval $\{0,1\}$

$$u(\tau) = f(\tau, x(\tau), x^{\cdot}(\tau) \quad (2)$$

with the initial condition $x(0) = \alpha$ (3)

or boundary conditions $x(0) = \alpha \quad x(1) = \beta$ (4)

The OCP can be reduced to a mathematical programming problem using either parametrization or discretization techniques to reduce the OCP to mathematical programming.

The optimal control problems had been studied through many works [4-9]. Their exact solution is not always exists, so numerical algorithms is the way to solve them. Numerical approaches for treating (OCPs) are greatly vary in their techniques and complexity, for example, the method of successive approximations based on Pontryagin's maximum principle is described in [10]. In [11] both direct and indirect methods are used to solve a viral marking model with optimal control. The direct and indirect methods are also utilized for treating (OCP) in Growth theory in [12]. Further application of indirect method is in fluid flow which is control problem in a two dimensional [13]. Also an optimal control problem with time delayed is solved in [14] based on Pontryagin's maximum principle. Closed form approximate solution was adopted by [15] for solving linear quadratic (OCP) with the aid of pontryagin's maximum principle. In addition, estimate solution of Crip (OCP) is presented [16] based on Euler-Lagrange conditions for more words on numerical solution of (OCPs) can be found in [17-19]. Basic spline functions are One of the popular basis functions which can be applied in many fields such as in solid state physics [20].

In this paper, two algorithms are considered to solve the problem described by Eqns. (1-4) based on B-spline functions to find an approximation solution to problem Eqns. 1 and 2 with the condition Eq. 3 and the second algorithm is to solve problem Eqns. 1-2 with the boundary conditions Eq. 4. The (OCP) is then converted to non–linear optimizations problem using state parameterization technique. The basic spline or B-spline in the mathematical subfield of numerical analysis is a spline function that has minimal support with respect to a specified degree and in computer aided design as well as computer graphic the spline function are represented as linear combination of B-spline with a set of control points. The basic spline is generalization of Bernstein polynomials with specific control points. Bernstein polynomials are an important basis functions that can be utilized in approximate the solution in various areas of mathematics.

2- The Goal and Organization of the Article

The first purpose of this paper is to discuss the state parameterization and show how it can be utilized in systematic way. The second purpose is to present the reformulation method of the optimal control problem into a mathematical programming problem with the help of basic spline functions. The third purpose is to derive an explicit algorithm for approximating the performance index.

For all of these objectives a numerical method to solve the special optimal control problem, named linear quadratic optimal control problem (LQOCP), by directly converting it into a mathematical programming problem. To this end the state parameterization method is applied based on the Bernstein polynomials or B-spline with specific control points, therefore the OCP is converted into a mathematical programming problem which can be solved simply . The advantages of this numerical method are: there is no need to integrate the system state; the OCP is converted into a small mathematical programming problem.

The organization of this paper is: in section, the basic formulation of B-spline is described then the relationship between Bernstein polynomials and basic spline is devoted in section 3. Section 4 reports our methods by considering two algorithms and illustrates the accuracy of the proposed two algorithms by giving some examples in section 5. Some conclusions are listed in section 6.

3- The Definition of Basic spline [21]

Suppose that an infinite set of knots $\{\tau_i\}$ is prescribed as

$$\ldots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \ldots \tag{5}$$

Then, the higher order of B-spline can be generating depending on the set of knots Eq. 5, in the following way

$$Bs_{i,n}(\tau) = (\tau - \tau_i)\frac{Bs_{i,n-1}(\tau)}{(\tau_{i+n} - \tau_i)} + (\tau_{i+n+1} - \tau)\frac{Bs_{i+1,n-1}(\tau)}{(\tau_{i+n+1} - \tau_{i+1})} \quad \text{where } n \geq 1 \tag{6}$$

Particular cases for the B-spline basic when $\tau = (\tau_0, \tau_1)$ that is $\tau_{-3}, \tau_{-2}, \tau_{-1}$ tend to word to τ_0 and τ_2, τ_3, τ_4 tend to word τ_1 if $\tau_0 = 0$ and $\tau_1 = 1$ then Eq. 6 will be

$$Bs_{i,n}(\tau) = \tau Bs_{i,n-1}(\tau) + (1-\tau)\, Bs_{i+1,n-1}(\tau) \quad \text{for } -k \leq i \leq 0 \text{ and } k=1,2,3 \tag{7}$$

4- Reduction B-spline basis functions to Bernstein polynomials

The basic spline is considered as generalization of Bernstein polynomials and they can be shared in most of their geometric and analytic properties. This fact is illustrated through this section.

Consider the following parametric knot values τ_1

$$\tau_i = \begin{cases} 0 & i < n+1 \\ i-n & n+1 \leq i \leq n \\ 1 & i > n \end{cases}$$

where $\tau \epsilon \{0, 1\}$ and $i \in \{0, 2n+1\}$, this means that

$$[\tau_0 \quad \tau_1 \quad \tau_2 \ldots \quad \tau_n \quad \tau_{n+1} \quad \tau_{2n+1}] = [0 \quad 0 \quad 0 \ldots 0 \quad 1 \ldots 1 \quad 1] \tag{8}$$

Now, for $n = 1$, one can obtain the first order B-spline using Eq. 6

$$Bs_{i1}(\tau)=\begin{cases}\dfrac{\tau-\tau_i}{\tau_{i+1}-\tau_i} & \tau\in[\tau_i,\tau_{i+1})\\ \dfrac{\tau_{i+2}-\tau}{\tau_{i+2}-\tau_{i+1}} & \tau\in[\tau_{i+1}.\tau_{i+2})\end{cases}$$

with the aid of Eq. 8, we can get the knot points $\tau_0,\tau_1,\tau_2,\tau_3$

$$[\tau_0\ \ \tau_1\ \ \tau_2\ \ \tau_3]=[0\quad 0\quad 1\quad 1\,] \qquad (9)$$

Hence

$$Bs_i^1(\tau)=\beta_0B_0^1(\tau)+\beta_1B_1^1(\tau) \qquad (10)$$

where

$$B_{01}(\tau)=\begin{cases}\dfrac{\tau-\tau_0}{\tau_1-\tau_0} & \tau\in[\tau_0,\tau_1)\\ \dfrac{\tau_2-\tau}{\tau_2-\tau_1} & \tau\in[\tau_1.\tau_2)\end{cases} \qquad (11)$$

and

$$B_{11}(\tau)=\begin{cases}\dfrac{\tau-\tau_1}{\tau_2-\tau_1} & \tau\in[\tau_1,\tau_2)\\ \dfrac{\tau_3-\tau}{\tau_3-\tau_1} & \tau\in[\tau_2.\tau_3)\end{cases} \qquad (12)$$

Using Eq. 9 in Eqns. 10 and 11, yields

$B_{01}(\tau)=1-\tau$ and $B_{11}(\tau)=\tau$

Therefore $Bs_{0.1}(\tau)=\beta_0(1-\tau)+\beta_1(\tau)$ (13)

Eq. 13 represents Bernstein polynomials of the first order

Similarly, one can prove that B-spline of nth order is Bernstein polynomials of order n with the parametric knot points

$$[\tau_0\ \ \tau_1\ \ \tau_2\ \dots\ \ \tau_{n+1}]=[0\quad 0\quad 0\dots\ 0\,] \qquad (14)$$

and $[\tau_{n+2}\ \ \tau_{n+3}\dots\tau_{2n}\ \ \tau_{2n+1}]=\ \ [1\ \ 1\ \ \dots\ 1\ \ 1]$ (15)

In this case the B-spline function can be written as:

$$Bs_{i1}(\tau)=\begin{bmatrix}\dfrac{\tau-\tau_i}{\tau_{i+1}-\tau_i} & \tau\in[\tau_{\mathrm{i}},\tau_{\mathrm{i+1}})\\ \dfrac{\tau_{i+2}-\tau}{\tau_{i+2}-\tau_{i+1}} & \tau\in[\tau_{\mathrm{i+1}},\tau_{\mathrm{i+2}})\\ \dfrac{\tau_{i+3}-\tau}{\tau_{i+3}-\tau_{i+2}} & \tau\in[\tau_{\mathrm{i+2}},\tau_{\mathrm{i+3}})\\ \mathrm{M} & \\ \dfrac{\tau_{i+n+1}-\tau}{\tau_{i+n+1}-\tau_{i+n}} & \tau\in[\tau_{\mathrm{i+n}},\tau_{\mathrm{i}+n+1})\end{bmatrix}$$

where

$B_{01}(\tau) = (1-\tau)^n$

$B_{11}(\tau) = \tau(1-\tau)^{(n-1)}$

$B_{12}(\tau) = \tau^2(1-\tau)^{(n-2)}$

M

$B_{nn}(\tau) = \tau^n$

which is Bernstein polynomials of order n and can be rewritten as

$$Bs_{in}(\tau) = B_{in}(\tau) = \sum_{i=0}^{n} \beta_i B_{in}(\tau)$$

where $B_{in}(\tau) = \binom{n}{i} \tau^i (1-\tau)^{n-i} \quad 0 \le i \le n$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \tau \in [0, 1)$

5- The Principle of the State Parameterization with B-spline

The algorithm $(SP - BS)_1$

To obtain an optimal performance value J(.) for problem (1-2) with Eq. 4, can follow These steps

Step 1: choose $\in > 0$

Step 2: let n=1, put $x_1(\tau) = a_0 Bs_{01}(\tau) + a_1 Bs_{11}(\tau)$

where $a_0 = x_0 = \alpha$ and $a_1 = x_1 = \beta$

$u_1(\tau) = f(\tau, x_1(\tau), \dot{x}_1(\tau))$, set $\sigma_1 = J(x_1(.))$

Step 3: set $n = 2$, $x_2(\tau) = a_0 Bs_{02}(\tau) + a_1 Bs_{12}(\tau) + a_2 Bs_{22}(\tau)$

where $a_0 = x_0(0)$ and $a_1 = x_1(1)$

$u_2(\tau) = f(\tau, x_2(\tau), \dot{x}_2(\tau))$, set $\sigma_2 = J(x^2(.))$

Step 4: set $n \longrightarrow n+2$, put $x_n(\tau) = a_0 Bs_{0n}(\tau) + a_1 Bs_{nn}(\tau) + \sum_{k=1}^{n-1} a_k Bs_{nk}(\tau)$

where $a_0 = x_0$ and $a_n = x_1$

$u_n(\tau) = f(\tau, x_n(\tau), \dot{x}_n(\tau))$ set $\sigma^n = J(x_n(.))$

Step 5: If $|\sigma_n - \sigma_{n-1}| > \in$ then go to step 3, otherwise , stop

The algorithm $(SP - BS)_2$

In order to solve problem (1-2) with Eq. 3, the following step3 are considered

Step1: let $n = 1$, state with approximate $x_1(\tau) = a_0 Bs_{01}(\tau) + a_1 Bs_{11}(\tau)$

where $a_0 = x_0 = \alpha$

$u_1(\tau) = f(\tau, x_1(\tau), \dot{x}_1(\tau))$set $\sigma^1 = J(x_1(.))$

Step 3: set $n = 2, x_2(\tau) = a_0 Bs_{02}(\tau) + a_1 Bs_{12}(\tau) + a_2 Bs_{22}(\tau)$

where $a_0 = x_0(0)$

$u_2(\tau) = f(\tau, x_2(\tau), \dot{x}_2(\tau))$, set $\sigma^2 = J(x_2(.))$

Step 4: set $n \longrightarrow n+2$, put $x_n(\tau) = a_0 Bs_{0n}(\tau) + a_1 Bs_{nn}(\tau) + \sum_{k=1}^{n-1} a_k Bs_{nk}(\tau)$

where $a_0 = x_0$

$u_n(\tau) = f(\tau, x_n(\tau), \dot{x}_n(\tau))$ set $\sigma^n = J(x_n(.))$

Step 5: If $|\sigma_n - \sigma_{n-1}| > \epsilon$ then go to step 3, otherwise stop

6- Application Examples

The effectiveness properties of the proposed technique are illustrated through the following given example.

Example (1)

The proposed method in this example is applied to the following problem

$$J = \int_0^1 (x(\tau)^2 + u(\tau)^2) dt \quad (16)$$

subject to $u(\tau) = \dot{x}(\tau)$ (17)

and boundary conditions $x(0) = 0, x(1) = 0.5$ (18)

By using algorithm$(\mathrm{SP} - \mathrm{BS})_1$ an approximate solution $x^1(\tau)$ is considered as initial approximation $x_1(\tau) = a_0 Bs_{01}(\tau) + a_1 Bs_{11}(\tau)$ (19)

Using the conditions in Eq. 18, one can get the parameters a_0 and a_1 as $a_0 = 0$ and $a_1 = \frac{1}{2}$

Hence, $x_1(\tau) = \frac{1}{2}\tau$ (20)

Using Eq. 17 to get $u_1(\tau) = \frac{1}{2}$ (21)

Then put Eqns. 20 and 21 into Eq. 16, yields

$$J = \int_0^1 \left(\frac{1}{4}\tau^2 + \frac{1}{4}\right) d\tau = 0.333333$$

The second approximation to $x_2(\tau), u_2(\tau)$ and the corresponding optimum value of J is given by

$$x_2(\tau) = a_1\left(2\tau - 2\tau^2\right) + \frac{1}{2}\tau^2$$

$u_2(\tau) = 2a_1 - 4a_1\tau + \tau$

$J = \frac{22a_1^2}{15} - \frac{17a_1}{30} + \frac{23}{60}$

The value $a_1 = \frac{17}{88}$ is minimize J,

This leads to $J = 0.328598$

The third approximation to $x_3(\tau), u_3(\tau)$ and the corresponding optimum value of J is given by

$$x_3(\tau) = \left(3a_1 - 3a_2 + \frac{1}{2}\right)\tau^3 + (3a_2 - 6a_1)\tau^2 + 3a_2\tau$$

$$u_3(\tau) = \left(9a_1 - 9a_2 + \frac{3}{2}\right)\tau^2 + (6a_2 - 12a_1)\tau + 3a_2$$

$$J = \frac{9a_1^2}{7} + \frac{51a_2a_1}{70} - \frac{4a_1}{7} + \frac{9a_2^2}{7} - \frac{29a_2^2}{35} + \frac{17}{35}$$

The value $a_1 = \frac{202}{1419}$

$a_2 = \frac{400}{1419}$ is minimize J,

This leads to $J = 0.328259$

The above results can be compared with the following exact solution

$$x(\tau) = A\,(e^{\tau} - e^{-\tau}),\quad u(\tau) = A(e^{\tau} + e^{-\tau})$$

where $A = \frac{e}{2(e^2-1)}$

and $J_{exact} = 0.3282588214$

By using algorithm $(\mathrm{SP} - \mathrm{BS})_1$, the stopping criteria $(|J_{n+1} - J_n| \leq 1 \times 10^{-6})$ is satisfied after three iterations that is when n=3 and the value of the performance index $J = 0.328259$ is obtained.

Example (2)

The objective of the second problem is following quadratic optimal control problem

$$\min J = \frac{1}{2}\int_0^1 (x(\tau)^2 + u(\tau)^2)dt \qquad (22)$$

$$\text{subject to } \dot{x}(\tau) = x(\tau) + u(\tau) \qquad (23)$$

$$\text{with initial condition } x(0) = 0 \qquad (24)$$

Here, Eqns. 23 and 24 with Eq. 22 is solved using algorithm $(\mathrm{SP} - \mathrm{BS})_2$, the results of approximate are summarized as follows

$x_1(\tau) = (1 - \tau) + a_1\tau$

$u_1(\tau) = a_1(1 + \tau) - \tau$

$J(a_1) = J(0.25) = 0.25$

Second approximate

$x_2(\tau) = (1 - 2\tau + \tau^2) + a_1(2\tau - 2\tau^2) + a_2\tau^2$

$\dot{x}_2(\tau) = -2 + 2\tau + a_1(2 - 4\tau) + 2a_2\tau$

$u_2(\tau) = \tau^2 - 2a_1\tau^2 + a_2\tau^2 - 1 + 2a_1 - 2a_1\tau + 2a_2\tau$

$J(a_1, a_2) = J\left(\frac{70}{187}, \frac{53}{187}\right) = 0.194296$

Third approximate

$x_3(\tau) = (1-3\,\tau + 3\tau^2 - \tau^3) + a_1(3\,\tau + 6\tau^2 - 3\tau^3) + a_2(3\tau^2 - 3\tau^3) + a_3\tau^3$

$\dot{x}_3(\tau) = (-3 + 6\tau - 3\tau^2) + a_1(3 + 12\tau - 9\tau^2) + a_2(6\tau - 9\tau^2) + 3a_3\tau^2$

$u_3(\tau) = (-2 + 3\tau + 3\tau^3) + a_1(3 - 9\,\tau + 3\tau^2 + 3\tau^3) + a_2(\,6\,\tau - 6\tau^2 + 3\tau^3) + a_3(\tau^3 + 3\tau^2)$

$J(a_1, a_2, a_3) = J(\frac{5589}{10264}, \frac{1957}{5132}, \frac{1447}{5132}) = 0.192932$

The obtained approximated results can be compared with the following actual solution

$$x(\tau) = 0.010039\, e^{\sqrt{2}e^{\sqrt{2}\tau}} + 0.989961\sqrt{2}e^{-\sqrt{2}\tau},$$

$$u(\tau) = 0.010039(\sqrt{2}+1)e^{\sqrt{2}e^{\sqrt{2}\tau}} - 0.989961(\sqrt{2}-1)\sqrt{2}e^{-\sqrt{2}\tau},$$

and $J_{exact} = 0.192932$

By using algorithm$(\mathrm{SP}-\mathrm{BS})_2$, the stopping criteria$(|J_{n+1} - J_n| \leq 1 \times 10^{-6})$ is satisfied after three iterations that is when n=3 and the value of the performance index $J = 0.192932$is obtained.

7- Conclusion

An accurate algorithm for solving optimal control problem governed by ordinary differential equation with the both initial condition or boundary condition is proposed in this paper based on the state parameterization technique. The idea of this approach is to approximate the state variables by a basic spline functions and the control variables are determined from the state equations. Examples are included to confirm the efficiency of the algorithm The following points are concluded:

- There is no need to integrate the system state equation.
- Three is few number of unknown parameters
- The system state and the conditions are satisfied directly.

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