Journal of Nature, Life and Applied Sciences
Volume (5), Issue (4): 30 Dec 2021
P: 98 - 110

ISSN: 2522-3356

مجلة العلوم الطبيعية والحياتية والتطبيقية
المجلد (5)، العدد (4): 30 ديسمبر 2021م
ص: 98 - 110

WEAK and CO-WEAK BAER MODULES

Eaman Al-Khouja
Magd Alfakhory
Faculty of Science || Al-Baath University || Syria
Hamza Hakmi
Faculty of Science || Damascus University || Syria

Abstract: The object of this paper is study the notions of weak Baer and weak Rickart rings and modules. We obtained many characterizations of weak Rickart rings and provide their properties. Relations ship between a weak Rickart (weak Baer) module and its endomorphism ring are studied. We proved that a weak Baer module with no infinite set of nonzero orthogonal idempotent elements in its endomorphism ring is precisely a Baer module. In addition, the endomorphism ring of a semi-projective weak Rickart module is semi-potent and the endomorphism ring of a semi-injective coweak Rickart module is semi-potent. Furthermore, we show that a free module is weak Baer if and only if its endomorphism ring is left weak Baer.

Keywords: weak Baer ring, weak Baer module, weak Rickart ring, weak Rickart module, endomorphism ring, Annihilator.

مـودولات بييـر الضعيفـة والضعيفـة المرافقـة

إيمان الخوجة
مجد الفاخوري
كلية العلوم || جامعة البعث || سوريا
حمزة حاكمي
كلية العلوم || جامعة دمشق || سوريا

المستخلص: الهدف من هذا البحث هو دراسة مفاهيم حلقات ومودولات بـيير الضعيفة وريكارت الضعيفة. لقد حصلنا على العديد من توصيفات حلقات ريكارت الضعيفة وقدمنا خصائصها. تمت دراسة العلاقة بين مودولات ريكارت الضعيفة (بيير الضعيفة) وحلقة الإندومورفيزمات الخاصة بها. لقد أثبتنا أن مودولات بيير الضعيفة التي تملك مجموعة غير منتهية من العناصر الجامدة المتعامدة غير الصفرية في حلقة الإندومورفيزمات الخاصة بها هي على وجه التحديد مودولات بيير. بالإضافة إلى ذلك، فإن حلقة الإندومورفيزمات الخاصة بمودولات ريكارت الضعيفة نصف الإسقاطية هي حلقة شبه جامدة وحلقة الإندومورفيزمات الخاصة بمودولات ريكارت المرافقة نصف الأفقية هي حلقة شبه جامدة. علاوة على ذلك، فقد بينا أن المودولات الحرة هي مودولات بيير الضعيفة إذا وفقط إذا كانت حلقة الإندومورفيزمات الخاصة بها هي حلقة بيير ضعيفة يسارية.

الكلمات المفتاحية: حلقة بيير الضعيفة، مودول بيير الضعيف، حلقة ريكارت الضعيفة، مودول ريكارت الضعيف، حلقة الإندومورفيزمات، العادم.

1. Introduction

It is considered that Kaplansky is the first who defined the concept of Baer ring in [9], where a ring R is said to be a Baer ring if every right (left) annihilator of any nonempty subset of R is generated by an idempotent as right (left) ideal. It is clear that these two notions are right-left symmetric. Following [3], a ring R is said to be a right (left) Rickart ring if every right (left) annihilator of any signal element of R is generated by an idempotent as right (left) ideal, which is equivalent, to every a principal right (left) ideal of R is projective; i.e., R is a right (left) $p.p-$ring [4] and [5]. It is well-known that the notion of Rickart rings is not left-right symmetric. The concept of Baer rings was extended by Rizvi-Roman [13] to the general module theoretic setting. A module M is said to be a Baer module if the right annihilator in M of every nonempty subset of $S = End_R(M)$ is generated by an idempotent of S, which is equivalent, to every left annihilator in S of any submodule of M is generated by an idempotent of S

In section 2, we study a weak Rickart rings and provide some characterizations and investigate its properties. We have proved that the endomorphism ring S of an $R-$module M is right weak Rickart ring if and only if for every $\alpha \in S$ with $Ker(\alpha) \neq 0$, $Ker(\alpha)$ contains a non-zero direct summand of M. Also, it is proved that the endomorphism ring S of an $R-$module M is left weak Rickart if and only if for every $\alpha \in S$ with $Im(\alpha) \neq M$, $Im(\alpha)$ contained in a direct summand of $K \neq M$ of M.

In section 3, it is proved that, if M is a semi-projective retractable module, then the endomorphism ring of M is a left weak Rickart ring if and only if M is a weak Rickart module and if M is a semi-injective co-retractable module, then the endomorphism ring of M is right weak Rickart if and only if M is a coweak Rickart module. In addition to, the endomorphism ring of a semi-projective weak Rickart module is semi-potent and the endomorphism ring of a semi-injective coweak Rickart module is semi-potent. In section 4, we study a weak Baer modules and we obtained many characterizations and conclude some of its properties. It is proved that the endomorphism ring of a weak Baer module is a left weak Baer ring. Also, we prove that a co-retractable module M is weak Baer if and only if every proper submodule of M contained in a direct summand $N \neq M$ of M. Also, We prove that a weak Baer module with no infinite set of nonzero orthogonal idempotent in its endomorphism ring is precisely a Baer module. In addition, a free module is weak Baer if and only if its endomorphism ring is left weak Baer.

Throughout this paper, R is associative ring with an unity element and M is an unital right $R-$module. For a right $R-$module M, $S = End_R(M)$ will denote the endomorphism ring of M. For $\alpha \in S$, $Ker(\alpha)$ and $Im(\alpha)$ stand for the kernel and the image of α, respectively. Also, for any nonempty subset I of S;

we denote of right annihilator of I in M by $r_M(I) = \{m : m \in M;\ I \cdot m = 0\}$ and the right annihilator of I in S $r_S(I) = \{\alpha : \alpha \in S;\ I \cdot \alpha = 0\}$. For any nonempty subset N of M, we denote of right annihilator of N in R by

$$r_R(N) = \{r : r \in R;\ N \cdot r = 0\}$$

and the left annihilator of N in S by $\lambda_S(N) = \{\alpha : \alpha \in S;\ \alpha(N) = 0\}$. For any element a of a ring R, we denote of left (right) annihilator of a in R by

$$\lambda(a) = \{x : x \in R;\ xa = 0\},\ r(a) = \{x : x \in R;\ ax = 0\}$$

2. Weak Rickart Rings.

Recall that a ring R is a left (right) Ricakrt ring if for every $a \in R$, $Ra = Re$ $(aR = eR)$ for some idempotent $e \in R$, [5]. In this section we introduce and study the notion of right (left) weak Rickart rings and investigate some of its properties. We start with the following Lemma:

Lemma 2.1. For any ring R the following conditions are equivalent:

1- For every $a \in R$ with $r(a) \neq 0$ there exists a non-zero idempotent $e \in R$ such that $e \in r(a)$.
2- For every $a \in R$ with $r(a) \neq 0$, $a = ae$ for some idempotent $1 \neq e \in R$.
3- For every $a \in R$ with $r(a) \neq 0$, $Ra \subseteq Rf$ where $1 \neq f^2 = f \in R$.
4- For every $a \in R$ with $r(a) \neq 0$, $g = (1-a)g$ where $0 \neq g^2 = g \in R$.
5- For every $a \in R$ with $r(a) \neq 0$, $eR \subseteq (1-a)R$ where $0 \neq e^2 = e \in R$

Proof. Obvious.

We say that a ring R is a right weak Rickart ring if it satisfies the conditions of Lemma 2.1. Similarly, we define a left weak Rickart ring. Also, we say that a ring R is a weak Rickart ring if R is a right and left weak Rickart ring. It is clear that every right (left) Rickart ring is a right (left) weak Rickart ring. Also, we get the following characterization:

Lemma 2.2. Let R be a ring without non-zero nilpotent elements. Then the following conditions are equivalent:

1- R is a weak Rickart ring.
2- R is a right weak Rickart ring.
3- R is a left weak Rickart ring.

Proof. It is clear, because for a ring R without non-zero nilpotent elements $r(a) = \lambda(a)$ for all $a \in R$.

A right order Q in a ring R is a subring of R such that every element of R has the form ab^{-1} for some $a, b \in Q$. Similarly, we define a left order in R.

Lemma 2.3. Let Q be a right order in a ring R. Then:

1- $r_R(ab^{-1}) = br_Q(a)R$ for every $a, b \in Q$.
2- If Q is a right weak Rickart ring, then R is a right weak Rickart ring.

Proof. (1). It is clear that $br_Q(a)R \subseteq r_R(ab^{-1})$.

Let $c, d^{-1} \in r_R(ab^{-1})$ where $c, d \in Q$. Since Q is a right order ring in R, we can write $b^{-1}c = ts^{-1}$ for some $t, s \in Q$. Then $t \in r_Q(a)$ and so

$$cd^{-1} = bts^{-1}d^{-1} \in br_Q(a)R.$$

This shows that $r_R(ab^{-1}) \subseteq br_Q(a)R$.

(2). Let $a, b^{-1} \in R$, $r_R(ab^{-1}) \neq 0$ where $a, b \in Q$, by (1) we have

$$r_R(ab^{-1}) = br_Q(a)R$$

and $r_Q(a) \neq 0$, so by assumption there exist an idempotent $0 \neq e \in Q$ such that $e \in r_Q(a)$, so $beb^{-1} \in R$ is a nonzero idempotent and

$$beb^{-1} \in br_Q(a)R = r_R(ab^{-1})$$

Theorem 2.4. For every $R-$module M with $S = End_R(M)$ the following are equivalent:

1- S is a right weak Rickart ring.

2- For every $\alpha \in S$ with $Ker(\alpha) \neq 0$, $Ker(\alpha)$ contains a nonzero direct summand of M.

3- For every $\alpha \in S$ with $Ker(\alpha) \neq 0$ there exist an idempotent $1 \neq g \in S$ such that $\alpha = \alpha g$.

Proof. $(1) \Rightarrow (2)$. Let $\alpha \in S$ with $Ker(\alpha) \neq 0$, then there are $g_1, g_2 \in S$, $g_1 \neq g_2$ such that $\alpha g_1 = \alpha g_2$. This shows that $r_S(\alpha) \neq 0$ by (1) there exists an idempotent $0 \neq e \in S$, $e \in r_S(\alpha)$, so $Im(e) \subseteq Ker(\alpha)$ and $Im(e) \neq 0$ is a direct summand of M.

$(2) \Rightarrow (1)$. Let $\alpha \in S$ with $Ker(\alpha) \neq 0$. Then by (2) $e(M) \subseteq Ker(\alpha)$ for some $e^2 = e \in S$. $e \neq 0$. So $\alpha e = 0$ and $\alpha = \alpha(1-e)$ where $1-e \neq 1$ is an idempotent of S.

$(3) \Rightarrow (1)$. Let $\alpha \in S$, $r_S(\alpha) \neq 0$. Then $\alpha\lambda = 0 = \alpha 0$ for some $0 \neq \lambda \in S$. This shows that α is not monomorphism, by (3) $\alpha = \alpha e$ for some $1 \neq e^2 = e \in S$. So $\alpha(1-e) = 0$ where $1-e \in S$ is a nonzero idempotent of S and so $1-e \in r_S(\alpha)$.

Theorem 2.5. For every $R-$module M with $S = End_R(M)$ the following are equivalent:

1- S is a left weak Rickart ring.

2- For every $\alpha \in S$ with $Im(\alpha) \neq M$, $Im(\alpha)$ contained in a direct summand $K \neq M$ of M.

3- For every $\alpha \in S$ with $Im(\alpha) \neq M$, $\alpha = g\alpha$ for some idempotent $1 \neq g \in S$.

Proof. The proof follows dually to the Theorem 2.4.

Lemma 2.6. Let M be an $R-$module, $\alpha \in S = End_R(M)$. If S is a right weak Rickart ring, then:

1- α is monomorphism if and only if $Ker(\alpha)$ is small in M.

2- If $Im(1-\alpha)$ is small in M, then α is an unit.

Proof. (1) $(\Rightarrow)$. Is obvious. $(\Leftarrow)$. Assume that $Ker(\alpha) \neq 0$ by Theorem 2.4 $e(M) \subseteq Ker(\alpha)$ for some $0 \neq e^2 = e \in S$. Since $Ker(\alpha)$ is small in M, $e(M)$ is small in M so $e = 0$ a contradiction.

(2).Suppose that $Im(1-\alpha)$ is small in M. Since $Ker(\alpha)\subseteq Im(1-\alpha)$, $Ker(\alpha)$ is also small in M so by (1) α is monomorphism. On the other hand, $M=Im(\alpha)$, because $M=Im(\alpha)+Im(1-\alpha)$, thus α is an unit.

Lemma 2.7. Let M be an $R-$module, $\alpha\in S=End_R(M)$. If S is a left weak Rickart ring, then:

1- α is an epimorphism if and only if $Im(\alpha)$ is large in M.
2- If $Ker(1-\alpha)$ is large in M, then α is an unit.

Proof. The proof follows dually to the Lemma 2.6.

Let P be an projective $R-$module, $\lambda\in S=End_R(P)$, it is well-known that the submodule $Im(\lambda)$ is small in P if and only if the right ideal λS is small in S [13, Proposition 1.1].

Also, if Q is an injective $R-$module and $\lambda\in S=End_R(Q)$, then the sub-module $Ker(\lambda)$ is large in Q if and only if the left ideal $S\lambda$ is small in S [11, Proposition 1, P.102].

Corollary 2.8. Let M be an $R-$module, $\alpha\in S=End_R(M)$. Then:

I – If M is projective and S is a left weak Rickart ring, then the submodule $Im(\alpha)$ is large in M if and only if the right ideal αS is large in S.

II – If M is injective and S is a right weak Rickart ring, then the submodule $Ker(\alpha)$ is small in M if and only if the left ideal $S\alpha$ is small in S.

Proof. (I) $(\Rightarrow)$. Suppose that $Im(\alpha)$ is large in M, then by Lemma 2.7 α is an epimorphism. Let I be a right ideal of S such that $\alpha S\cap I=0$. Since M is projective, for every $\lambda\in I$ there exists $\mu\in S$ such that $\lambda=\alpha\mu\in\alpha S$, so $I\subseteq\alpha S$. Thus $I=\alpha S\cap I=0$. This shows that αS is large in S.

$(\Leftarrow)$. Suppose that $Im(\alpha)$ is not large in M, then $Im(\alpha)\neq M$ by Theorem 2.5 $Im(\alpha)$ contained in a direct summand $N\neq M$ of M, so $Im(\alpha)\subseteq N=e(M)$ for some idempotent $1\neq e\in S$. Since M is projective, $\alpha S\subseteq eS$. Thus

$$\alpha S\cap(1-e)S=0$$

Since αS is large in S, $(1-e)S=0$ so $e=1$ a contradiction. This implies that $Im(\alpha)$ is large in M. (II) The proof follows dually to (I).

Lemma 2.9. Let M be an $R-$module, $S=End_R(M)$.

I. If S is a right weak Rickart ring, then for every $\alpha\in S$, $Ker(\alpha)\neq 0$ the following holds:
 1- $Im(1-\alpha)$ contains a nonzero direct summand of M.
 2- $Ker(1-\alpha)$ contained in a direct summand $N\neq M$ of M.

II. If S is a left weak Rickart ring, then for every $\alpha\in S$, $Im(\alpha)\neq M$ the following holds:
 1- $Im(1-\alpha)$ contains a nonzero direct summand of M.
 2- $Ker(1-\alpha)$ is contained in a direct summand $N\neq M$ of M.

Proof. (I) Let $\alpha\in S$, $Ker(\alpha)\neq 0$. By Theorem 2.4 $\alpha=\alpha\sigma$ where $\sigma\in S$ is an idempotent and $\sigma\neq 1$. Since $\alpha(1-\sigma)=0$ and $1=\alpha+(1-\alpha)$,

$$1-\sigma=(1-\sigma)(1-\alpha)(1-\sigma).$$

1- Let $g=(1-\alpha)(1-\sigma)$. Then $g\in S$ is a nonzero idempotent element and $Im(g)\subseteq Im(1-\alpha)$ where $Im(g)\neq 0$ is a direct summand of M.

2- Let $f=(1-\sigma)(1-\alpha)$. Then $0\neq f\in S$ is an idempotent and

$$Ker(1-\alpha)\subseteq Ker(f)=Im(1-f)$$

where $1\neq 1-f\in S$ is an idempotent and so, $Im(1-f)\neq M$ is a direct summand of M.

II – The proof follows dually to the (I).

Example. (1) Recall that an $R-$module M is regular [16], if for every $m\in M$, $m=mf(m)$ for some $f\in Hom_R(M,R)$. If M is a regular module, then $S=End_R(M)$ is a right weak Rickart ring. Because for every $\alpha\in S$ with $Ker(\alpha)\neq 0$, $Ker(\alpha)$ contains a direct summand of M and by Theorem 2.4, S is right weak Rickart.

(2) Call that a ring R is an I_0-ring [8] (Semi-potent ring [17]), if for every right (left) ideal of R which not contained in the Jacobson radical $J(R)$ of R, contains a nonzero idempotent. It is clear that every I_0-ring with zero Jacobson radical is a weak Rickart ring.

(3) Call that a ring R is regular [7] if for every $a\in R$, $a=axa$ for some $x\in R$. It is clear that every regular ring is a weak Rickart ring.

3. Weak Rickart Modules.

Recall that an $R-$module M is a Rickart module [6], if the right annihilator in M of any signal element of $S=End_R(M)$ is generated by an idempotent of S, equivalently, $r_M(\alpha)=Ker(\alpha)$ is a direct summand of M for every $\alpha\in S$. Note that:

$$Ker(\alpha)=r_M(\alpha)=r_M(S\alpha).$$

Let M_R be a module and $S=End_R(M)$. We say that a module M is a weak Rickart module if for $\alpha\in S$ with $\lambda_S(Im(\alpha))\neq 0$, $\lambda_S(Im(\alpha))$ contains a nonzero idempotent of S. Similarly, we say that a module M is a coweak Rickart module if for $\alpha\in S$ with $r_M(\alpha)\neq 0$, $r_M(\alpha)$ contains a nonzero direct summand of M. It is clear that every Rickart module is coweak Rickart.

Corollary 3.1. For every $R-$module M with $S=End_R(M)$. Then the following statements are equivalent:

1- The module M is coweak Rickart.

2- For any $\alpha\in S$ with $Ker(\alpha)\neq 0$, $Ker(\alpha)$ contains a nonzero direct summand of M.

3- The ring S is a right weak Rickart ring.

Proof. $(1)\Leftrightarrow(2)$. Is obvious because $r_M(\alpha)=Ker(\alpha)$ for every $\alpha\in S$.

$(2)\Leftrightarrow(3)$. By Theorem 2.4.

Corollary 3.2. For every $R-$module M with $S=End_R(M)$. Then the following statements are equivalent:

1- The module M is weak Rickart.
2- For every $\alpha \in S$ with $Im(\alpha) \neq M$, $Im(\alpha)$ contained in a direct summand of $K \neq M$ of M.
3- The ring S is a left weak Rickart ring.

Proof. $(1) \Leftrightarrow (2)$. Is obvious because $\lambda_S(Im(\alpha)) = \lambda_S(\alpha)$ for every $\alpha \in S$. $(2) \Leftrightarrow (3)$. By Theorem 2.5.

Call that an $R-$module M is co-retractable [2], if for every proper sub-module N of M, $\lambda_S(N) \neq 0$. Also, Call that an $R-$module M is retractable [2], if $hom_R(M,N) \neq 0$ for every submodule $N \neq 0$ of M.

Let M be an $R-$module, $S = End_R(M)$. Following Wisbauer [14], a module M is called semi-injective if for every $\alpha \in S$, $S\alpha = \lambda_S(Ker(\alpha))$. Also, a module M is called semi-projective if for every $\alpha \in S$,

$$\alpha S = Hom_R(M, Im(\alpha))$$

It is clear that if M is a semi-injective co-retractable module, then $Ker(\alpha) = 0$ if and only if $S = S\alpha$ for every $\alpha \in S$. Also, if M is a semi-projective retractable module, then $Im(\alpha) = M$ if and only if $\alpha S = S$ for every $\alpha \in S$.

Proposition 3.3. Let M be a semi-injective co-retractable $R-$module. Then the following statements are equivalent:

1- The module M is coweak Rickart.
2- For any $\alpha \in S$ with $S\alpha \neq S$, $Ker(\alpha)$ contains a nonzero direct summand of M.
3- The ring S is a right weak Rickart ring.

Proof. Is obvious by Corollary 3.1 and our assumption.

Proposition 3.4. Let M be a semi-projective retractable $R-$module. Then the following statements are equivalent:

1- The module M is weak Rickart.
2- For any $\alpha \in S$ with $\alpha S \neq S$, $Im(\alpha)$ contained in a direct summand of $K \neq M$ of M.
3- The ring S is a left weak Rickart ring.

Proof. Is obvious by Corollary 3.2 and our assumption.

Let M_R be a module and $S = End_R(M)$, suppose that

$$\hat{\nabla} S = \{\alpha : \alpha \in S; Im(1-\beta\alpha) = M; \text{ for all } \beta \in S\}$$

$$\hat{\Delta} S = \{\alpha : \alpha \in S; Ker(1-\alpha\beta) = 0; \text{ for all } \beta \in S\}$$

In [8], it is proved that $J(S) \subseteq \hat{\nabla} S$, $J(S) \subseteq \hat{\Delta} S$ and if M is semi-projective,

$J(S) = \hat{\nabla} S$ [8, Lemma 3.2]. Also, if M is semi-injective, $J(S) = \hat{\Delta} S$ [8, Lemma 3.7].

Proposition 3.5. Let M be a weak Rickart $R-$module, $S = End_R(M)$. The following hold:

1- For every $\alpha \in S$, $\alpha \notin \hat{\nabla} S$, αS contains a nonzero idempotent of S.

2- If M is semi-projective, then S is semi-potent.

Proof. 1 – Let $\alpha \notin \hat{\nabla} S$, then there exists $\sigma \in S$ such that $Im(1-\sigma\alpha) \neq M$, by Corollary 3.2 $Im(1-\alpha\sigma) \subseteq Ker(e)$ for some idempotent $0 \neq e \in S$. Since $e(1-\alpha\sigma) = 0$, $\sigma e = (\sigma e)\alpha(\sigma e)$. Let $g = \alpha(\sigma e)$, then g is a nonzero idempotent of S and $g = \alpha(\sigma e) \in \alpha S$.

2 – Follows from (1), hence $J(S) = \hat{\nabla} S$.

Proposition 3.6. Let M be a coweak Rickart $R-$module, $S = End_R(M)$. The following hold:

1- For every $\alpha \in S$, $\alpha \notin \hat{\Delta} S$, αS contains a nonzero idempotent of S.
2- If M is semi-injective, then S is semi-potent.

Proof. The proof follows dually to the Proposition 2.4.

4. Weak Baer Rings and Modules.

We say that a ring R is a right (left) weak Baer ring if for every nonempty subset $I \subseteq R$, $r(I) \neq 0$ ($\lambda(I) \neq 0$)there exists an idempotent $0 \neq e \in R$ such that $e \in r(I)$ ($e \in \lambda(I)$). Also, we say that a ring R is a weak Baer ring if R is a right and left weak Baer ring. It is clear that every Baer ring is a right (left) weak Baer ring.

Lemma 4.1. For every ring R the following statements are equivalent:

1- R is a right (left) weak Baer ring.
2- For any nonempty subset $J \subseteq R$, $\lambda(J) \neq R$ ($r(J) \neq R$) there exists an idempotent $1 \neq f \in R$ such that $\lambda(J) \subseteq Rf$ ($r(J) \subseteq fR$.
3- For any nonempty subset $I \subseteq R$, $r(J) \neq 0$ ($\lambda(I) \neq 0$) there exists an idempotent $1 \neq f \in R$ such that $a = af$ ($a = fa$) for all $a \in I$.

Proof. Is obvious.

Lemma 4.2. For any ring R the following holds:

1- If R is a right (left) Rickart ring, then R is a left (right) weak Baer ring.
2- If R is a Rickart ring, then R is a weak Baer ring.

Proof. 1 – Assume that R is a right Rickart ring. Let I be a nonempty subset of R with $r(I) \neq 0$. Then $Ia = 0$ for some $0 \neq a \in R$, by assumption $\lambda(a) = Re$ for some idempotent $1 \neq e \in R$. Since $\lambda(r(I)) \subseteq \lambda(a) = Re$, $\lambda(r(I))(1-e) = 0$ so $1-e \in r(\lambda(r(I))) = r(I)$ where $1-e \in R$ is a nonzero idempotent.

2 – Is obvious by (1).

Recall that an $R-$module M is a Baer module [12], if for every sub-module N of M, $\lambda_S(N) = Se$ for some $e^2 = e \in S$. We say that a module M is a weak Baer module

if for any submodule N of M with $\lambda_S(N) \neq 0$, $\lambda_S(N)$ contains a nonzero idempotent of $S = End_R(M)$. It is clear that every Baer module is a weak Baer module.

Lemma 4.3. Let M be an $R-$module, $S = End_R(M)$. Then the following statements are equivalent:

1- The module M is a weak Baer module.

2- For every submodule N of M with $\lambda_S(N) \neq 0$ there exists a direct summand $M_1 \neq M$ of M such that $N \subseteq M_1$.

3- For every left ideal I of S with $r_M(I) \neq M$ there exists a direct summand $K \neq M$ of M such that $r_M(I) \subseteq K$.

Proof. $(1) \Rightarrow (2)$. Let N be a submodule of M with $\lambda_S(N) \neq 0$, then $e \in \lambda_S(N)$ for some idempotent $0 \neq e \in S$. Since $e(N) = 0$, $N \subseteq Ker(e)$ and $Ker(e) \neq M$ is a direct summand of M.

$(2) \Rightarrow (3)$. Let I be a left ideal of S with $r_M(I) \neq M$. Since $r_M(I)$ is a sub-module of M such that $\lambda_S(r_M(I)) \neq 0$, so by (2) $r_M(I) \subseteq K$ for some direct summand $K \neq M$ of M.

$(3) \Rightarrow (1)$. Let N be a submodule of M with $\lambda_S(N) \neq 0$. Since $\lambda_S(N)$ is a left ideal of S and $r_M(\lambda_S(N)) \neq M$, so by (3) $r_M(\lambda_S(N)) \subseteq K$ for some direct summand $K \neq M$ of M. Thus $r_M(\lambda_S(N)) \subseteq e(M)$ where $e: M \to K$ the projection onto K. Since $K \neq M$, $1 \neq e \in S$ is an idempotent and so

$$1 - e \in \lambda_S(e) = \lambda_S(e(M)) \subseteq \lambda_S(r_M(\lambda_S(N))) = \lambda_S(N)$$

where $1 - e \in S$ is a nonzero idempotent. This proves $(3) \Rightarrow (1)$.

Proposition 4.4. Let M be a weak Baer $R-$module. Then $S = End_R(M)$ is a left weak Baer ring.

Proof. Let I be a nonempty subset of S with $\lambda_S(I) \neq 0$. Assume that $N = \sum_{\alpha \in I} Im(\alpha)$ then N is a submodule of M such that $\lambda_S(N) = \lambda_S(I) \neq 0$ by assumption $e \in \lambda_S(N) = \lambda_S(I)$ for some idempotent $0 \neq e \in S$. Therefore S is a left weak Baer ring.

Theorem 4.5. Let M be a co-retractable $R-$module, $S = End_R(M)$. Then the following statements are equivalent:

1- The module M is a weak Baer module.

2- Every submodule $U \neq M$ of M, U contained in a direct summand $N \neq M$ of M.

Proof. $(1) \Rightarrow (2)$. Let $U \neq M$ be a submodules of M. Since M is co-retractable, $\lambda_S(U) \neq 0$ by assumption there exists an idempotent $0 \neq e \in S$ such that $e \in \lambda_S(U)$. Since $e(U) = 0$, $U \subseteq Ker(e) = Im(1-e)$, where $Im(1-e) \neq M$ is a direct summand of M.

$(2) \Rightarrow (1)$. Let N be a submodule of M with $\lambda_S(N) \neq 0$, then $N \neq M$. By assumption $N \subseteq K$ for some direct summand $K \neq M$ of M. Thus $M = K \oplus K_0$ for some submodule $K_0 \neq 0$ of M. Let $e: M \to K_0$ be the projection onto K_0, Then $0 \neq e \in S$ is idempotent and $e \in \lambda_S(N)$, because $e(N) \subseteq e(K) = 0$.

Therefore M is a weak Baer module.

Theorem 4.6. Let F be a right free $R-$module and $S = End_R(F)$. Then the following statements are equivalent:

1- F is a weak Baer module.

2- S is a left weak Baer ring.

Proof. $(1) \Rightarrow (2)$. By Proposition 4.4. $(2) \Rightarrow (1)$. Let A be a submodule of F with $\lambda_S(A) \neq 0$. Since F is free, $A = \sum_{\alpha \in I} Im(\alpha)$ for some nonempty subset I of S. So $\lambda_S(I) = \lambda_S(A) \neq 0$, by assumption $\lambda_S(I)$ contains a idempotent $e \in S$, $e \neq 0$ thus $e \in \lambda_S(A)$. This shows that F is a weak Baer module.

Lemma 4.7. [10, Theorem 7.55] For every right $R-$module M, $End_R(M)$ contains no infinite set of nonzero orthogonal idempotent elements if and only if $End_R(M)$ has DCC on direct summand left ideals if and only if M has ACC on direct summand.

Theorem 4.8. Let M be a right $R-$module, $S = End_R(M)$ has no infinite set of orthogonal idempotent elements. Then the following are equivalent:

1- M is a Baer module.

2- M is a weak Baer module.

3- M is a Rickart module.

Proof. $(1) \Rightarrow (2)$. Is obvious. $(2) \Rightarrow (1)$. Let N be a submodule of M. If $\lambda_S(N) = 0$, then $\lambda_S(N)$ is a direct summand of S. Suppose that $\lambda_S(N) \neq 0$, then by assumption $\lambda_S(N)$ contains a nonzero idempotent of S. From Lemma 4.7, the hypothesis on S amounts to the fact that direct summand of S satisfy DCC. Among all nonzero idempotent in $\lambda_S(N)$, choose $e \in \lambda_S(N)$ with $S(1-e) = \lambda_S(e) = \lambda_S(e(M))$ minimal. We will prove that

$$\lambda_S(N) \cap \lambda_S(e(M)) = 0$$

Suppose that $\lambda_S(N) \cap \lambda_S(e(M)) \neq 0$, then

$$\lambda_S(N + e(M)) = \lambda_S(N) \cap \lambda_S(e(M)) \neq 0.$$

Since M is a weak Baer module, there exists a nonzero idempotent $f \in S$ such that $f \in \lambda_S(N) \cap \lambda_S(e(M))$, so $f \in \lambda_S(e(M))$.

Since $fe = 0$, $e' = e + (1-e)f \in S$ is a nonzero idempotent and $e \neq e'$. In addition, $e'e = e$, so $eS \subseteq e'S$. Thus

$$\lambda_S(e') = \lambda_S(e'S) \subseteq \lambda_S(eS) = \lambda_S(e)$$

which implies $\lambda_S(e'(M)) \subseteq \lambda_S(e(M))$. Moreover, $\lambda_S(e'(M)) \neq \lambda_S(e(M))$, because if $\lambda_S(e'(M)) = \lambda_S(e(M))$, then $f \in \lambda_S(e'(M))$, so $fe' = 0$ and $f(1-e)f = 0$, i.e. $f = 0$ a contradiction. This contradicts the choice of e.

Finally, since $e \in \lambda_S(N)$, $Se \subseteq \lambda_S(N)$. On the other hand, for any $\alpha \in \lambda_S(N)$,

$$\alpha(1-e) = \alpha - \alpha e \in \lambda_S(N) \cap \lambda_S(e(M)) = 0$$

so $\alpha = \alpha e \in Se$, Thus $\lambda_S(N) \subseteq Se$ and so $\lambda_S(N) = Se$. Therefor M is a Baer module. $(1) \Leftrightarrow (3)$ By [5, Theorem 4.5].

We say that a module M is a coweak Baer module if for any left ideal I of $S = End_R(M)$ with $r_M(I) \neq 0$, $r_M(I)$ contains a nonzero direct summand of M. It is clear that every Baer module is cowek Baer. Recall that a module M is quasi-retractable [12], if $hom_R(M, r_M(I)) \neq 0$ for every left ideal I of $S = End_R(M)$ such that $r_M(I) \neq 0$. Obviously, every retractable module is quasi-retractable.

Theorem 4.9. Let M be a right $R-$module, $S = End_R(M)$. Then the following statements are equivalent:

1- M is a coweak Baer module.
2- For every left ideal I of S with $r_M(I) \neq 0$, $I \subseteq Se$ for some idempotent $1 \neq e \in S$.
3- For every submodule N of M with $\lambda_S(N) \neq S$, $\lambda_S(N) \subseteq Se$ for some idempotent $1 \neq e \in S$.
4- A ring S is a right weak Baer ring and M is a quasi-retractable module.

Proof. $(1) \Rightarrow (2)$. Let I be a left ideal of S such that $r_M(I) \neq 0$, by assumption $r_M(I)$ contains a direct summand $N \neq 0$ of M, so $N = e(M)$ for some idempotent $0 \neq e \in S$. Since $N \subseteq r_M(I)$;

$$I \subseteq \lambda_S(r_M(I)) \subseteq \lambda_S(e(M)) = \lambda_S(eS) = S(1-e)$$

where $1 \neq 1-e \in S$ is an idempotent.

$(2) \Rightarrow (3)$. Is trivial, because $\lambda_S(N)$ is a left ideal of S with $r_M(\lambda_S(N)) \neq 0$.

$(3) \Rightarrow (4)$. Let I be a nonempty subset of S such that $\lambda_S(I) \neq S$. Then $r_M(\lambda_S(N)) \neq 0$ is a submodule of M and $\lambda_S(r_M(\lambda_S(I))) \neq S$, so by assumption $\lambda_S(I) \subseteq Se$ for some idempotent $1 \neq e \in S$, because $\lambda_S(I) = \lambda_S(r_M(\lambda_S(I)))$. Therefore by Lemma 3.1, S is a right weak Baer ring. Let J be a left ideal of S with $r_M(J) \neq 0$. Since $r_M(J)$ is a submodule of M such that $\lambda_S(r_M(J)) \neq S$, $\lambda_S(r_M(J)) \subseteq Sf$ for some idempotent $1 \neq f \in S$ by (3). Thus:

$$Im(1-f) = Ker(f) \subseteq \cap_{\lambda \in S} Ker(\lambda f) = r_M(Sf) = r_M(\lambda_S(r_M(J))) = r_M(J)$$

so $0 \neq 1-f \in hom_R(M, r_M(J))$. This shows that M is quasi-retractable.

$(4) \Rightarrow (1)$. Let I be a left ideal of S such that $r_M(I) \neq 0$, then by assumption $hom_R(M, r_M(I)) \neq 0$. Thus $Im(\lambda) \subseteq r_M(I) = \cap_{\varphi \in I} Ker(\varphi)$ for some $0 \neq \lambda \in S$. Thus $\varphi\lambda = 0$ for every $\varphi \in I$, so $I\lambda = 0$ and so $\lambda \in r_S(I)$, this shows that $r_S(I) \neq 0$. Since S is a right weak Baer ring, there exists an idempotent $0 \neq e \in S$ such that $e \in r_S(I)$. So $Im(e) \subseteq \cap_{\alpha \in I} Ker(\alpha) = r_M(I)$ and $Im(e) \neq 0$ is a direct summand of M, this shows that M is a coweak Baer module.

It is clear that every free module is retractable. From this fact and Theorem 4.9 the following is derived:

Corollary 4.10. A free module F_R is a coweak Baer module if and only if $End_R(F)$ is a right weak Baer ring.

Lemma 4.11. [1, Lemma 2.2] Let V and U be submodules of a projective module P. Then $P = V + U$ if and only if $S = \hat{V} + \hat{U}$ where $S = End_R(P)$ and $\hat{V} = hom_R(P,V)$, $\hat{U} = hom_R(P,U)$.

Lemma 4.12. Every projective module $P_R \neq 0$ with $J(P) = 0$ is retractable.

Proof. Let A be a nonzero submodule of P, then there exists a maximal submodule M of P such that $A \not\subset M$. By Lemma 4.11, $1 = \alpha + \beta$ for some $\alpha, \beta \in S = End_R(P)$ and $Im(\alpha) \subseteq A$, $Im(\beta) \subseteq M$. It is clear that $\alpha \neq 0$, because if $\alpha = 0$, then $P = M$ a contradiction, so P is retractable.

Using Theorem 4.9 and Lemma 4.12, we obtain the following result.

Corollary 4.13. A projective module $P_R \neq 0$ with $J(P) = 0$ is a coweak Baer module if and only if $End_R(P)$ is a right weak Baer ring.

5. Recommendations.

we recommend studying ring which the endomorphism ring of all modules above it is a weak and co-weak Baer ring.

References

[1] G. Azumaya, $F-$Semi-Perfect Modules, J. Algebra. 136 (1991), p.73 – 85.

[2] B. Amini, and M. Ersha, and H. Sharif, Co-Retractable Modules, J. Aust. Math. Soc. 86(3) (2009), p. 289 – 304.

[3] E. P. Armendariz, A Note On Extensions of Baer and P.P.-Rings, J. Aust. Math. Soc. 18 (1974), p. 470 – 473.

[4] S. Endo, A Note on p.p. Rings, Nagoya Math. J. 17(1960), p. 167 – 170.

[5] G. Lee, and Tariq Rizvi, and Cosmin, Roman, Rickart Modules, Communications in Algebra. 38 (11) (2010), p. 4005 – 4027.

[6] G. Lee, and S. T. Rizvi, Direct Sums of Quasi-Baer Modules, J. Algebra. 456(2016), p. 76 – 92.

[7] K. R. Goodearl, Von Neumann Regular Rings, Pitman, London. (1979).

[8] H. Hamza, (Co)Retractability and (Co)Semi-potency, Korean J. Math. 25 (4) (2017), p. 587 – 606.

[9] Kaplansky, Rings of Operators, New York-Amsterdam, (1968).

[10] T. Y. Lam, Lectures on Modules and Rings, Springer Verlag. (1999).

[11] J. Lambek, Lectures on Rings and Modules, Blaisdell, Mass. (1966).

[12] S. T. Rizvi, and C. S. Roman, Baer and Quasi-Baer Modules, Comm. Algebra. 32(1) (2004), p. 103 – 123.

[13] R. Ware, Endomorphism Rings of Projective Modules, Trans. Amer. Math. Soc. 155 (1971), p. 233 – 256.

[14] R. Wisbauer, Foundations of Modules and Rings Theory, Philadelphia: Gordon and Breach. (1991).

[15] Thoraya Abdelwhab, Xiaoyan Yang. "Modules whose Endomorphism Rings Are Right Rickart", Asian Research Journal of Mathematics, 13(2) (2019), p. 1-14.

[16] J. Zelmanowitz, Regular Modules, Trans. Amer. Math. Soc. 163 (1972), p. 340 – 355.

[17] Y. Zhou, On (Semi)Regularity and the Total of Rings and Modules, J. Algebra. 322(2009), p. 562 – 578.