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Numerical Spline Method for Simulation of Stochastic Differential Equations systems

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Abstract: In this paper, numerical spline method is presented with collocation two parameters for solving systems of multi-dimensional stochastic differential equations (SDEs). Multi-Wiener's time-continuous process is simulated as a discrete process, and then the mean-square stability of proposed method when applied to a system of two-dimensional linear SDEs is studied. The study shows that the method is mean-square stability and third-order convergent when applied to a system of linear and nonlinear SDEs. Moreover, the effectiveness of our method was tested by solving two test linear and non-linear problems. The numerical results show that the accuracy and applicability of the proposed method are worthy of attention.

Keywords: Stochastic Differential Equations Systems, Multi-Wiener Process, Spline Collocation Polynomial, Mean-Square Stability, Mean-Square Convergence.

طريقة شرائحية عددية لمحاكاة حل نظم من المعادلات التفاضلية العشوائية

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المستخلص: نقدم في هذا البحث طريقة شرائحية عددية مع وسيطي تجميع لحل نظم من المعادلات التفاضلية العشوائية متعددة الأبعاد. تمت محاكاة عملية وينر العشوائية متعددة الأبعاد المستمرة مع الزمن كعملية منفصلة، ثم دراسة الاستقرار العشوائي بمتوسط المربعات للطريقة عندما تُطَبقْ لحل نظم من المعادلات التفاضلية العشوائية الخطية من البعد الثاني. تبين الدراسة أن الطريقة تكون مستقرة ومتقاربة بمتوسط المربعات من المرتبة الثالثة عندما يتم تطبيقها لحل نظم من المعادلات التفاضلية العشوائية خطية وغير خطية.

وقد تم اختبار فعالية الطريقة المقترحة بحل مسألتي اختبار الأولى خطية والثانية غير خطية، وتشير النتائج العددية إلى فعالية وكفاءة الطريقة الشرائحية المقترحة وأن النتائج الحاصلة جديرة بالاهتمام.

الكلمات المفتاحية: نظم معادلات تفاضلية عشوائية، عملية وينر متعددة الأبعاد، كثيرة حدود شرائحية، الاستقرار بمتوسط المربعات، التقارب العددي.

1. Introduction.

Until recently, many studies ignored random effects models, due to the great difficulty of finding solutions to these models. But now, stochastic differential equations play an important and prominent role in multiple fields after the tremendous technological development in industrial and scientific applications and their wide uses in modeling random phenomena, and they occur in the system of differential equations that are affected by random noise, and we mention, for example, in the economics (finance, Interest rate, stock prices), population growth, physics (fluid particles, thermal noise), control science (signal processing, tuning, filtering), medicine (number of cancer cells, number of people with epidemic disease), biology, and mechanics , Etc. Unfortunately, obtaining analytical solutions for such models is not available in most cases. Therefore, researchers are interested in developing numerical methods to simulate analytical solutions with discrete solutions.

Unfortunately, in many cases analytic solutions are not available for systems of stochastic differential equations, for these reasons, searchers numerical methods are developed to solve such systems [2, 3, 4], developments of Rung Kutta methods from various stages [5,9,10] discussed the numerical solutions of SDEs. Linda et al [6] introduced a comparison of three different stochastic population models with regard to persistence time. Baccouch, B. Johnson [7] develop a high-order discontinuous Galerkin method for solving SDEs of Itô type driven by Wiener processes. Bayram et al [8] studied the Euler-Maruyama (EM) and Milstein methods, and then to numerical solution is approximated using Monte Carlo simulation for each method. Haghighi & Rößler [1], constructed a class of split-step double balanced methods for the approximation of solutions of autonomous stiff SDEs. In this paper, spline collocation method for solving systems of multi-dimensional SDEs is proposed. In Section 2, the solution method is formulated. Mean-square stability (MS-stability) and error estimation for solution method are presented in Section 3-4. Numerical results are reported in Section 5.

Definition 1. (*m*-Dimensional Wiener Processes).

A stochastic process $(W(t))_{t\geq 0}$, $W: R_+ \to R^m$ is an *m*-dimensional Wiener process (also called *m*-dimensional Brownian motion), if it satisfies the following; for all i = 1,2,...m the stochastic process $(W_i(t))_{t\geq 0}$ is a one-dimensional Wiener process with $W(t)=(W_1(t),...,W_m(t))^T$. Let $W(t)=(W_1(t),...,W_m(t))^T$ be an *m*-dimensional Wiener process and a system of diagonal stochastic differential equations can be formulated:

$$\begin{bmatrix} dX_{1,t} \\ dX_{2,t} \\ \vdots \\ dX_{m,t} \end{bmatrix} = \begin{bmatrix} f_1(t,\bar{X}) \\ f_2(t,\bar{X}) \\ \vdots \\ f_m(t,\bar{X}) \end{bmatrix} dt + \begin{bmatrix} g_1(t,\bar{X}) & 0 & \cdots & 0 \\ 0 & g_2(t,\bar{X}) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_m(t,\bar{X}) \end{bmatrix} \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ \vdots \\ dW_{m,t} \end{bmatrix}, \qquad (1)$$

where $\overline{X} = (X_{1,t}, X_{2,t}, \ldots, X_{m,t})^T$.

2. Formulation of the Solution Method

A spline collocation technique is presented for solving a system of stochastic differential equations given in relation (1) by Itˆo's formula. Denoting $t_k = k\ dt, k = 0,1,\ldots,n$ the grid points of the uniform partition of [0, *T*] into subintervals $I_k = [t_k, t_{k+1}],\ k = 0,1,\ldots,n-1$, and $dt = T/n$ is the constant step size.

Let $X_i(t) \in \Re$, there is then a Hermite polynomial of degree at most third:

$$S_i(t) = \overline{\xi}^2(1+2\xi)S_{i,k} + \overline{\xi}^2\xi S_{i,k}^{[1]} + \xi^2(1+2\overline{\xi})S_{i,k+1} - \xi^2\overline{\xi}\ S_{i,k+1}^{[1]},$$

$$t \in [t_k, t_{k+1}], \quad k = 0,1,\ldots,n-1 \tag{2}$$

where

$$S_{i,\,k}^{[1]} = h\,dS_{i,k}(t_k),\ S_{i,k+1}^{[1]} = h\,dS_{i,k+1}(t_k),\ k = 0,1,\ldots,n-1,\ i = 1,2,\ldots,m$$

$\xi = (t - t_k)/dh,\quad \overline{\xi} = 1 - \xi \in [0,1]$, and $dS_{i,0}(0),\ S_{i,0}(0)$, initial values can be found from the starting conditions of the problem.

Differentiating the polynomial (2) with respect to t we get

$$dS_i(t) = 6(\overline{\xi}^2 - \overline{\xi})S_{i,k} + (3\overline{\xi}^2 - 2\overline{\xi})S_{i,k}^{[1]} + 6\,(\xi - \xi^2)S_{i,k+1} + (3\xi^2 - 2\xi)\ S_{i,k+1}^{[1]} \tag{3}$$

Generally, the grid determine s *n* spline polynomials $S_{i,k}(t), i = 1,2,\ldots,n$ are given as

$$S_i(t) = \begin{cases} S_{i,0}(t), & t \in [0,\ t_1], \\ \vdots & \\ S_{i,k}(t)\ , & t \in [t_k, t_{k+1}], \\ \vdots & \\ S_{i,n-1}(t), & t \in [t_{n-1}, T] \end{cases}$$

which fulfills the conditions

- $\mathbf{S}_i(t) \in \Re;\quad \mathbf{S}_i(t_k) \cong X_i(t_k),\ k = \mathbf{0,1,\ldots,}n$
- $S_{i,k}(t_k) = S_{i,k-1}(t_k)\ ,\quad k = 1,2,\ldots,n$
- $dS_{i,k}(t_k) = dS_{i,k-1}(t_k),\qquad k = 1,2,\ldots,\mathrm{n-1}$.

Let us know two collocation points

$$t_{k+z_j} = t_k + z_j\ dt, \qquad j = 1,2\ , \tag{4}$$

ino subintervals $I_k = [t_k, t_{k+1}],\ k = 0,1,\ldots, n-1,$ with two collocation parameters are given as

$$0 < z_1 < z_2 = 1 \tag{5}$$

Now by applying Hermite 's spline polynomials (2)-(3) with collocation points (4)-(5) into the system of stochastic differential equations (1), we get

$$
\begin{bmatrix} dS_1(t_{k+z_j}) \\ dS_2(t_{k+z_j}) \\ \vdots \\ dS_m(t_{k+z_j}) \end{bmatrix} = \begin{bmatrix} f_1(t_{k+z_j},\bar{S}_{k+z_j}) \\ f_2(t_{k+z_j},\bar{S}_{k+z_j}) \\ \vdots \\ f_2(t_{k+z_j},\bar{S}_{k+z_j}) \end{bmatrix} dt +
$$

$$
+\begin{bmatrix} g_1(t_{k+z_j},\bar{S}_{k+z_j}) & 0 & \cdots & 0 \\ 0 & g_2(t_{k+z_j},\bar{S}_{k+z_j}) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_m(t_{k+z_j},\bar{S}_{k+z_j}) \end{bmatrix} \begin{bmatrix} dW_1(\tau_k) \\ dW_2(\tau_k) \\ \vdots \\ dW_m(\tau_k) \end{bmatrix}, \quad (6)
$$

$$
j=1,2,\ k=0,1,\ldots,n-1,
$$

where

$\bar{S}_{k+z_j} = (S_1(t_{k+z_j}), S_2(t_{k+z_j}), \ldots, S_m(t_{k+z_j}))^T$ and $t_{k+z_j} \in]t_k, t_{k+1}]$, $j=1,2$.

The system (6) is rewritten in the matrix form, so we get the following iterative relationship:

$$
\bar{A}\hat{S}_{k+1} = \bar{B}\hat{S}_k + h\bar{F}_k + \bar{G}_k\bar{W}_k \ , \ k=0,1,\ldots,n-1 \ , \quad (7)
$$

where

$$
\bar{A} = \begin{bmatrix} A & O & \cdots & O \\ O & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B & O & \cdots & O \\ O & B & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & B \end{bmatrix}, \quad (8)
$$

$$
A = \begin{bmatrix} 6(z_1 - z_1^2) & 3z_1^2 - 2z_1) \\ 6(z_2 - z_2^2) & 3z_2^2 - 2z_2) \end{bmatrix}, \quad (9)
$$

$$
B = \begin{bmatrix} 6z_1(1-z_1) & 4z_1 - 3z_1^2 - 1) \\ 6z_2(1-z_2) & 4z_2 - 3z_2^2 - 1) \end{bmatrix}, \quad (10)
$$

$$
\bar{G}_k = \begin{bmatrix} G_{k,1} & \hat{O} \\ \hat{O} & G_{k,2} \end{bmatrix},
$$

$$
G_{k,j} = \begin{bmatrix} g_1(t_{k+z_j},\bar{S}_{k+z_j}) & 0 & \cdots & 0 \\ 0 & g_2(t_{k+z_j},\bar{S}_{k+z_j}) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_m(t_{k+z_j},\bar{S}_{k+z_j}) \end{bmatrix}, j=1,2,
$$

$$\hat{S}_{k+1}=\begin{bmatrix} S_{1,k+1} \\ S_{1,k+1}^{[1]} \\ S_{2,k+1} \\ S_{2,k+1}^{[1]} \\ \vdots \\ S_{m,k+1} \\ S_{m,k+1}^{[1]} \end{bmatrix},\ \hat{S}_k=\begin{bmatrix} S_{1,k} \\ S_{1,k}^{[1]} \\ S_{2,k} \\ S_{2,k}^{[1]} \\ \vdots \\ S_{m,k} \\ S_{m,k}^{[1]} \end{bmatrix},\ \overline{F}_k=\begin{bmatrix} f_1(t_{k+z_1},\overline{S}_{k+z_1}) \\ f_2(t_{k+z_1},\overline{S}_{k+z_1}) \\ \vdots \\ f_m(t_{k+z_1},\overline{S}_{k+z_1}) \\ f_1(t_{k+1},\overline{S}_{k+1}) \\ f_2(t_{k+1},\overline{S}_{k+1}) \\ \vdots \\ f_m(t_{k+1},\overline{S}_{k+1}) \end{bmatrix},\ \overline{W}=\begin{bmatrix} dW_{1,k} \\ dW_{2,k} \\ \vdots \\ dW_{m,k} \\ dW_{1,k} \\ dW_{2,k} \\ \vdots \\ dW_{m,k} \end{bmatrix}$$

Noting that O is 2×2 zero matrix , $\hat{O}$ is $m\times m$ zero matrix and $\overline{B}, \overline{A}$ are $2m\times 2m$ matrices, $dW_{i,k}=\sqrt{dt}\ \delta_{i,k}$, $h=dt$ step size and $\delta_{i,k}$ are N(0, 1) independent normally distributed numbers for i=1,2,...,m.

Notice that System (7) is always solvable for $0<z_1<z_2=1$, and the vector of unknowns $\hat{S}_{k+1}$ can be determined into subintervals $I_k=[t_k,t_{k+1}],\ k=0,1,\ldots,n-1,$ because the matrix A given by relation(9) always has an inverse where we have

$$Det(A)=\mathbf{6}z_1(\mathbf{1}-z_1)\neq\mathbf{0}.$$

3. M S-Stability of method for two-dimensional linear SD systems

Numerical MS- stability is studied by applying proposed method to test problem:

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix}=\begin{bmatrix} \lambda_1 X_t \\ \lambda_2 Y_t \end{bmatrix}dt+\begin{bmatrix} \mu_1 X_t & 0 \\ 0 & \mu_2 Y_t \end{bmatrix}\begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix} \tag{11}$$

with the following initial conditions

$$\begin{bmatrix} X(t_0=\mathbf{0}) \\ Y(t_0=\mathbf{0}) \end{bmatrix}=\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} \tag{12}$$

where $\lambda_i,\ \mu_i$ for i=1,2 real constants.

Applying the spline approximations (2)-(3) with two collocation points (4)-(5) into the problem (11)-(12), we obtain the linear system

$$\begin{bmatrix} dS_1(t_{k+z_j}) \\ dS_2(t_{k+z_j}) \end{bmatrix}=\begin{bmatrix} \lambda_1 S_1(t_{k+z_j}) \\ \lambda_2 S_2(t_{k+z_j}) \end{bmatrix}dt+\begin{bmatrix} \mu_1 S_1(t_{k+z_j}) & \mathbf{0} \\ \mathbf{0} & \mu_2 S_2(t_{k+z_j}) \end{bmatrix}\begin{bmatrix} dW_{1,k} \\ dW_{2,k} \end{bmatrix}, \tag{13}$$
$$j=\mathbf{1},\mathbf{2}\ ,\ \ k=\mathbf{0},\mathbf{1},\ldots,n-1$$

with the initial conditions $S_1(t_0=0)=X_0,\ S_2(t_0=0)=Y_0$,

where $t_{k+z_j}=t_k+z_j\ dt,\ j=\mathbf{1},\mathbf{2}$, $dW_{1,k}=\sqrt{dt}\ \delta_{1,k}$, $dW_{2,k}=\sqrt{dt}\ \delta_{2,k}$, $\delta_{1,k}$ *and* $\delta_{2,k}$ are N(0,1).

By applying spline approximations to the system (13), we have the following iterative system:

$$M_{\mathbf{1}}\overline{\overline{S}}_{k+\mathbf{1}} = M_{\mathbf{2}}\overline{\overline{S}}_{k}\ , \quad k=0,1,...,n\text{-}1 \tag{14}$$

$$M_1 = \begin{bmatrix} A-\lambda_1 h\, C-\mu_1\sqrt{h}\,\overline{C} & O \\ O & A-\lambda_2 h\, C-\mu_2\sqrt{h}\,\overline{C} \end{bmatrix},$$
$$M_2 = \begin{bmatrix} B-\lambda_1 h D-\mu_1\sqrt{h}\,\overline{D} & O \\ O & B-\lambda_2 h D-\mu_2\sqrt{h}\,\overline{D} \end{bmatrix}, \tag{15}$$

where the two matrices *A*, *B* are (9)-(10), and $\overline{C}, \overline{D}, C, D$ are given as follows:

$$C = \begin{bmatrix} z_1^2(3-2z_1) & z_1^2(z_1-1) \\ z_2^2(3-2z_2) & z_2^2(z_2-1) \end{bmatrix},\ D = \begin{bmatrix} 3z_1^2-2z_1^3-1 & -z_1(1-z_1)^2 \\ 3z_2^2-2z_2^3-1 & -z_2(1-z_2)^2 \end{bmatrix}, \tag{16}$$

$$\overline{C} = \begin{bmatrix} z_1^2(3-2z_1)\ \delta_{1,k} & z_1^2(z_1-1)\ \delta_{2,k} \\ z_2^2(3-2z_2)\ \delta_{1,k} & z_2^2(z_2-1)\ \delta_{2,k} \end{bmatrix},$$
$$\overline{D} = \begin{bmatrix} (3z_1^2-2z_1^3-1)\ \delta_{1,k} & -z_1(1-z_1)^2\ \delta_{2,k} \\ (3z_2^2-2z_2^3-1)\ \delta_{1,k} & -z_2(1-z_2)^2\ \delta_{2,k} \end{bmatrix}, \tag{17}$$

and two vectors

$$\overline{\overline{S}}_{k+1} = (S_{1,k+1}, S_{1,k+1}^{[1]}, S_{2,k+1}, S_{2,k+1}^{[1]})^T,\ \overline{\overline{S}}_{k} = (S_{1,k}, S_{1,k}^{[1]}, S_{2,k}, S_{2,k}^{[1]})^T.$$

Furthermore, we rewrite the recurring relation (14) as

$$\overline{\overline{S}}_{k+\mathbf{1}} = R(h,\lambda_{\mathbf{1}},\mu_{\mathbf{1}},\lambda_{\mathbf{2}},\mu_{\mathbf{2}})\overline{\overline{S}}_{k} \tag{18}$$

where

$$R(h,\lambda_{\mathbf{1}},\mu_{\mathbf{1}},\lambda_{\mathbf{2}},\mu_{\mathbf{2}}) = M_{\mathbf{1}}^{-\mathbf{1}}M_{\mathbf{2}}, \tag{19}$$

it is 4×4 a matrix. Note that the frequency relation (19) is important as a basis for defining the stochastic stability of our spline method applied to the test system (11).

And using the concept of the expected value with the mean of squares, we have:

$$E[dW_{i,k}]=0,\ E[dW_{i,k}^2]=h,\ E[\delta_{i,k}]=1,\ i=1,2$$

Thus, from the system (18), we obtain the iterative relation:

$$\overline{Y}_{k+1} = \overline{R}(h,\lambda_1,\mu_1,\lambda_2,\mu_2)\ \overline{Y}_k \tag{20}$$

where

$$\overline{Y}_{k+\mathbf{1}} = \begin{bmatrix} ES_{\mathbf{1},k+\mathbf{1}}^{\mathbf{2}} \\ E(S_{\mathbf{1},k+\mathbf{1}}^{[\mathbf{1}]})^{\mathbf{2}} \\ ES_{\mathbf{2},\ k+\mathbf{1}}^{\mathbf{2}} \\ E(S_{\mathbf{2},k+\mathbf{1}}^{[\mathbf{1}]})^{\mathbf{2}} \end{bmatrix},\ \overline{Y}_k = \begin{bmatrix} ES_{\mathbf{1},k}^{\mathbf{2}} \\ E(S_{\mathbf{1},k}^{[\mathbf{1}]})^{\mathbf{2}} \\ ES_{\mathbf{2},k}^{\mathbf{2}} \\ E(S_{\mathbf{2},k}^{[\mathbf{1}]})^{\mathbf{2}} \end{bmatrix}$$

The matrix $\overline{R}(h,\lambda_1,\mu_1,\lambda_2,\mu_2) = E[R^t(h,\lambda_1,\mu_1,\lambda_2,\mu_2)R(h,\lambda_1,\mu_1,\lambda_2,\mu_2)]$ is called MS-stability function of proposed spline method.

Definition 2.[2] If $\|.\|_\infty$ is vector norm on $\Re^4$ then a matrix norm $A=(a_{ij})$ on the set of real 4×4 matrices corresponding to the vector norms is defined in l_∞ as

$$\|A\|_\infty = \max_{\|x\|=1}\|Ax\| = \max_{1\le i\le 4}\sum_{j=1}^{4}|a_{ij}|$$

Definition 3. [2] The numerical method is said to be mean-square stable for $h,\lambda_1,\mu_1,\lambda_2,\mu_2$ if under matrix norm $\|.\|_\infty$ provided that $\|\overline{R}(h,\lambda_1,\mu_1,\lambda_2,\mu_2)\|<1$.

Without loss of generality, we put $\lambda=\lambda_1=\lambda_2, \mu=\mu_1=\mu_2$, and for real $p=\lambda h$ and $q=\mu\sqrt{h}$, the MS-stability function of this method is given by

$$\overline{R}(p,q)=E[R^t(h,\lambda,\mu)R(h,\lambda,\mu)].$$

Locating the Boundary of the MS-stability Region

The region Ω_{Spl} defined by

$$\Omega_{Spl}=\{(p,q)\ ;\ \|R(p,q)\|<1\ ,\mathrm{p,q}\in\Re\}$$

is called the MS-stability region of our spline method, , where $p=\lambda h$ and $q=\mu\sqrt{h}$. If collocation parameters $z_1=4/5, z_2=1$ this method possesses MS-stability region lies inside the shaded circle given in Fig. 1.

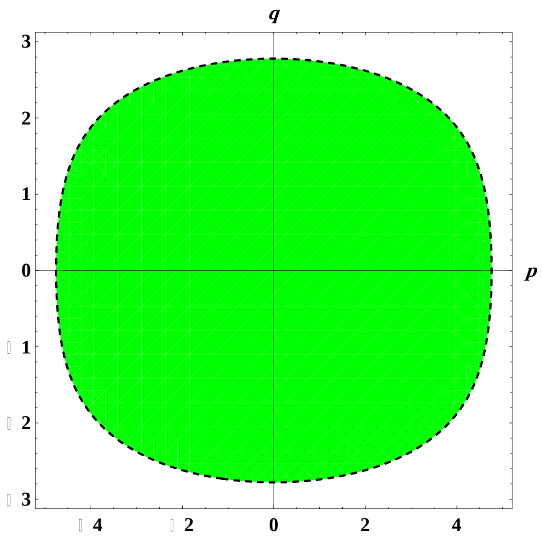

Fig. (1) MS- stability region of spline method

Corollary1: The spline method defined by the iterative relation (20) is MS-stable if $\|\overline{R}(p,q)\|<1$ for the collocation parameters $z_1=4/5, z_2=1$ and stepsize $h=0.99$.

Moreover, if the standard values $\lambda_1=2, \mu_1=-2,\ \lambda_2=\mathbf{2}, \mu_2=-\mathbf{2}$ are selected then the stability matrix for the method is given as follows:

$$\overline{R}(0.99,2,-2,2,-2)=\begin{bmatrix}0.72063 & 0.022743 & 0 & 0\\ 0.0446942 & 0.691573 & 0 & 0\\ 0 & 0 & 0.720630 & 0.022743\\ 0 & 0 & 0.0446942 & 0.691573\end{bmatrix},$$

so, $\| \overline{R}(0.99,2,-2,2,-2) \|_{\infty} = 0.743373$, we conclude according to definition 3 that our proposed method is SM-stable.

4. Error Estimation for solution Method

Definition 4 : [1] A discrete time approximation $\overline{\boldsymbol{S}}_{\boldsymbol{k}}$ is said to be mean-square convergent with order p>0 to the solution $\overline{\boldsymbol{X}}_{\boldsymbol{k}}$ of SDE (1) at time t_k if there exist two constants *C*>0 and $\delta_0 > 0$, such that $(E\,|\,\overline{X}_k - \overline{S}_k\,|^2)^{1/2} \le C\,h^p$ for each $h \in (0,\delta_0)$, especially, the constant *C* is independent of *h*.

Assuming that $X(t), Y(t) \in \Re$, the solution functions of a system of two equations (11)-(12), and $S_{1,k}, S_{2,k}$ the spline approximations for them, then we obtain the formula for the local truncated error of the method proposed by the system (13) as follows:

$$E\,|\,\overline{X}_k - \overline{S}_k\,|^2 = \begin{bmatrix} E\,|\,X\ \ (t_{k+1}) - S_1\,(t_{k+1})\,|^2 \\ E\,|\,(dX\,(t_{k+1})) - dS_1\,(t_{k+1})\,|^2 \\ E\,|\,Y\ \ (t_{k+1}) - S_2\,(t_{k+1})\,|^2 \\ E\,|\,(dY(t_{k+1})) - dS_2\,(t_{k+1})\,|^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5184} h^8 (2\alpha-1)^2 \,\|\, X^{(4)}(t_k)\,\|^2 \\ 0 \\ \frac{1}{5184} h^8 (2\alpha-1)^2 \,\|\, Y^{(4)}(t_k)\,\|^2 \\ 0 \end{bmatrix},$$

therefore, we find that

$$(E\,|\,\overline{X}_k - \overline{S}_k\,|^2)^{1/2} = \frac{1}{72}(2\alpha-1) M\ h^4, \tag{21}$$

where $M = Max\{\| X^{(4)}(t_k) \|, \| Y^{(4)}(t_k) \|\}$, and $\alpha \in (0,1)$ is stability parameter.

Furthermore, global truncated error is estimated at end of the interval, after *n* steps as follows

$$e(h) = C\,h^3,$$

where $C = \frac{1}{72}(2\alpha-1) M$.

Corollary 2: According to definition 4 our method is mean-square convergent with order third for any one uses the collocation parameters $\boldsymbol{z}_1 = 4/5, \boldsymbol{z}_2 = 1$, stepsize $\boldsymbol{h} = 0.99$.

5. Numerical Results

We test the effectiveness and efficiency of our method by applying it to the solution simulation of three test systems of stochastic differential equations, linear and nonlinear. To calculate the numerical convergence rate for the proposed spline method we will use the notations $e_k^N = E\,|\,S_k^N - S_{2k}^{2N}\,|$, *k*=1,...,*N*, where e_k^N indicates the mean absolute error at point $t_k = k\,h$ in [0,*T*], and S_k^N spline solution by using our methods, for step size *h*=*T*/*N*. The rate of numerical convergence is computed by

$$Rate_k = \frac{Ln(e_k^{N_1} / e_{2k}^{N_2})}{Ln(N_1 / N_2)},$$

where N_1=*N*, N_2=2*N*. All numerical results are obtained from computer program designed by *Mathematica*11.

Problem 1 [7]

Let our first test be the solution of the stiff linear stochastic differential system according to the following Ito formula:

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt + \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}\begin{pmatrix} X_t dW_{1,t} \\ Y_t dW_{2,t} \end{pmatrix}, \quad t \in [0,1],$$

with the initial conditions

$$X(0) = 1, \; Y(0) = 2.$$

The exact solution of this equation is given by

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = P \begin{pmatrix} \exp(\rho^+(t)) & 0 \\ 0 & \exp(\rho^-(t)) \end{pmatrix} P^{-1} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}, \quad P = P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\rho^{\pm}(t) = (a + \frac{1}{2}b^2 \pm a)t + bW(t).$$

This system is stiff if a is large, we choose the parameters as a= 100 and b= 0.5.In Fig.2 - 3, we plot, respectively, the spline solution by our method for X and Y in the time interval [0,1] with N = 2 ^ 7. We summarize in Table 1 the mean absolute error and the convergence rate in the spline solution for our method applied to find X and Y respectively, with stepsize h = 0.0625 in [0,1]. In Fig. 4 - 5, we plot the mean absolute error in the spline solution for X and Y with stepsize h = 0.0625 in [0,1].

Table (1) The mean absolute error and the convergence rate for Problem 1 by our method on uniform meshes having N=2 ^ 7, in [0,1].

t_k	Mean absolute error of X		Rate of convergence	Mean absolute error of Y		Rate of convergence
	e_k^N	e_{2k}^{2N}		e_k^N	e_{2k}^{2N}	
0.0625	0.00189964	0.00016373	3.53629	0.0004952	0.0000168	4.881478
0.125	0.00467074	0.00052474	3.15399	0.0003304	0.0000125	4.724563
0.1875	0.00739552	0.00019171	5.26964	0.0003258	0.0000114	4.829307
0.25	0.00042481	0.00002845	3.90009	0.0032737	0.0000951	5.10531
0.3125	0.00165938	0.00011311	3.87484	0.0046302	0.0002691	4.10484
0.375	0.00102205	0.00040274	4.66548	0.0035094	0.0001707	4.36091
0.4375	0.00567039	0.00035948	3.97946	0.0083339	0.0003371	4.62736
0.5	0.00534861	0.00042318	3.6598	0.0058793	0.0001968	4.90082
0.5625	0.00344965	0.00040929	3.07522	0.0006598	0.0000214	4.94549
0.625	0.00418766	0.00005399	6.27726	0.0074968	0.000125	5.89571
0.6875	0.00470726	0.00038984	3.59392	0.0007571	0.0000299	4.65937
0.75	0.00837453	0.00034640	4.59549	0.0007728	0.0000415	4.219023
0.8125	0.00204799	0.00031846	4.11174	0.0009227	0.0000536	4.105018
0.875	0.00230086	0.00017391	3.72570	0.0063864	0.0002348	4.76496
0.9375	0.00199561	0.00018614	3.42236	0.0024243	0.0001428	4.08550
1.	0.00434319	0.00052076	3.06006	0.0033437	0.0003372	3.81726

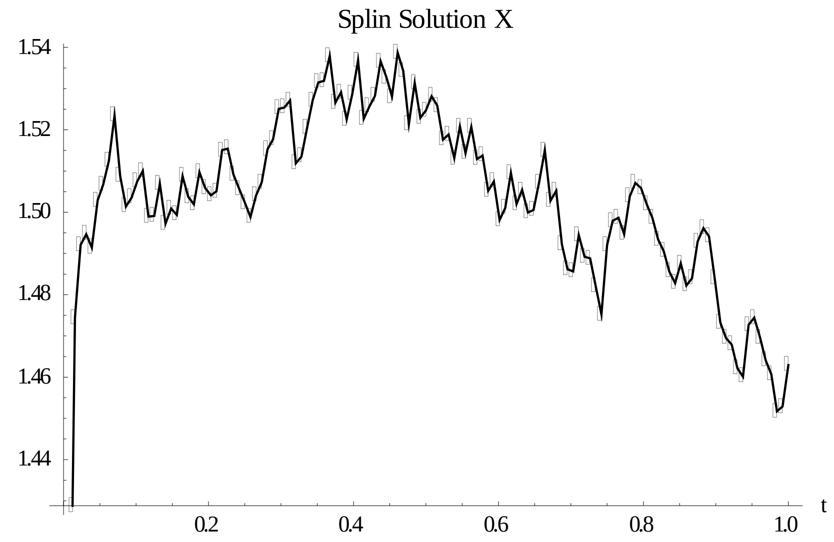

Fig. (2) Simulation of the spline solution (dashed curve) of *X*, with the exact solution (solid curve) for *N* = 2 ^ 7.

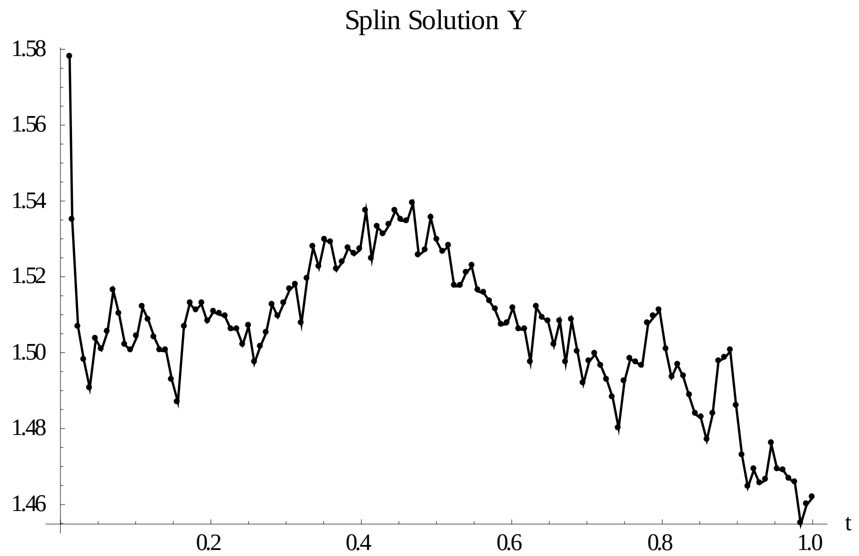

Fig. (3) Simulation of the spline solution (dashed curve)of *Y*, with the exact solution (solid curve) for *N* = 2 ^ 7.

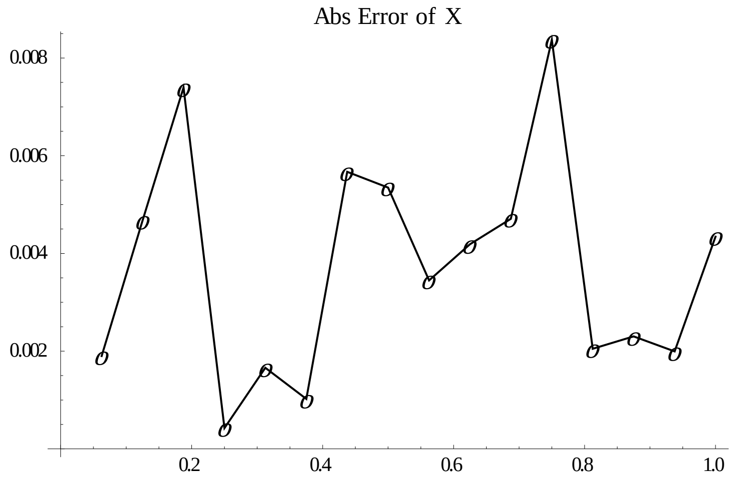

Fig (4) Mean absolute error in the proposed *X*-spline solution with *h* = 0.0625.

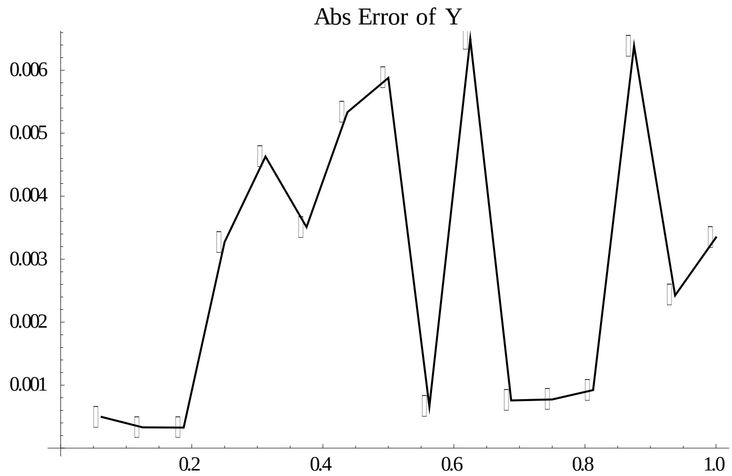

Fig (5) Mean absolute error in the proposed *Y*-spline solution with *h* = 0.0625.

Problem 2

The second is the biological system, we consider a two-species competition model with per capita birth and death rates given by

$$b_1(t)=0.84,\ \ b_2(t)=0.90 \text{ (birth rates)}$$

$$d_1(t)=0.40+0.01X(t)+0.022Y(t),\ d_2(t)=0.75+0.0067Y(t)+0.005X(t),$$

(death rates for both population types)

The resulting deterministic model is given as follows

$$\begin{cases} \dfrac{dX(t)}{dt}=[b_1(t)-d_1(t)]X(t) \\ \dfrac{dY(t)}{dt}=[b_2(t)-d_2(t)]Y(t) \end{cases},\ t\in[0,T] \tag{23}$$

with initial conditions $X(0) = 15$, $Y(0) = 15$.

The corresponding model of (23) is nonlinear stochastic differential system:

$$\begin{cases} \dfrac{dX(t)}{dt} = [b_1(t) - d_1(t)]\,X(t) + \sqrt{[b_1(t) + d_1(t)]X(t)}\,\dfrac{dW_1(t)}{dt}, \\ \dfrac{dY(t)}{dt} = [b_2(t) - d_2(t)]Y(t) + \sqrt{[b_2(t) + d_2(t)]Y(t)}\,\dfrac{dW_2(t)}{dt}, \end{cases}$$

with initial conditions $X(0) = 15$, $Y(0) = 15$.

In Figs.6-7, we draw, simulation of the spline solution for two population types X, and Y, respectively ,and in Fig. 8, we draw simulation of spline solution for the two types X, Y together, and in Fig. 9 ,we draw spline simulation of X-Y plane and all of these Figs. in [0,100], with $N = 2\wedge 7$.In Figs 10-11, we plot the mean absolute error in the spline solution for X and Y respectively, with $h = 0.0625$ in [0,100].In Fig. 12, the spline solution with the exact solution is plotted for $N = 2\wedge 7$ in [0,100].In Figs.13-14, we plot the numerical solution by the Range-Kutta method of second order[2] for X and Y and comparisons with the exact solution, with $h = 0.112$ in [0,28].

Table (2) Simulation of the spline solution of both types X and Y by our method for $N = 2\wedge 7$, in [0,100].

t_k	Spline solution by our method		Range-Kutta method of second order	
	X	Y	X	Y
0.0	15	15	15	15
3.125	14.008	13.8529	14.8401	12.5107
6.25	14.3995	14.1862	14.154	12.1837
9.375	14.2504	14.0041	11.6939	10.681
12.5	14.9743	13.3134	16.748	8.91756
15.625	15.0193	12.8704	21.2727	6.76912
18.75	14.5809	12.2507	22.0999	5.90549
21.875	15.9646	11.2563	27.1034	4.17056
25.0	17.9682	10.4333	32.1457	2.90323
28.125	19.4173	10.3487	35.6097	0.794911
31.25	20.3699	9.73913	-----	-----
34.375	21.3635	9.46446	-----	-----
37.5	21.688	9.5503	-----	-----
37.5	21.688	9.5503	-----	-----
43.75	24.5395	7.65855	-----	-----
46.875	26.0916	6.30343	-----	-----
50.	28.604	5.79438	-----	-----
53.125	30.7365	5.46369	-----	-----
56.25	31.8209	4.80047	-----	-----
56.25	31.8209	4.80047	-----	-----
62.5	34.5477	3.71191	-----	-----
65.625	35.9306	3.10666	-----	-----

t_k	Spline solution by our method		Range-Kutta method of second order	
	X	Y	X	Y
68.75	37.1074	2.67578	-----	-----
71.875	39.1055	1.91141	-----	-----
75.0	40.2974	1.55361	-----	-----
78.125	40.809	1.29919	-----	-----
81.25	40.4512	1.12202	-----	-----
84.375	40.8439	1.04651	-----	-----
87.5	42.3191	0.827401	-----	-----
90.625	42.848	0.743619	-----	-----
93.75	43.2406	0.646742	-----	-----
96.875	43.6822	0.518487	-----	-----
100.	43.5953	0.476432	-----	-----

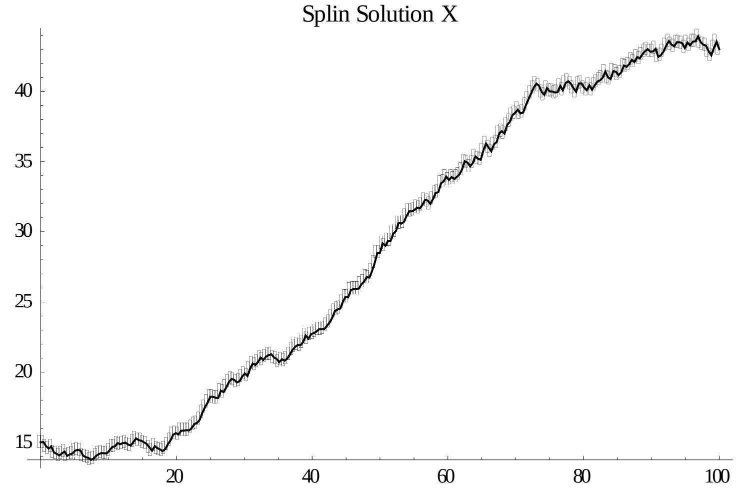

Fig. (6) Spline solution for the first population type X, for $N = 2^7$, $h = 25/32$.

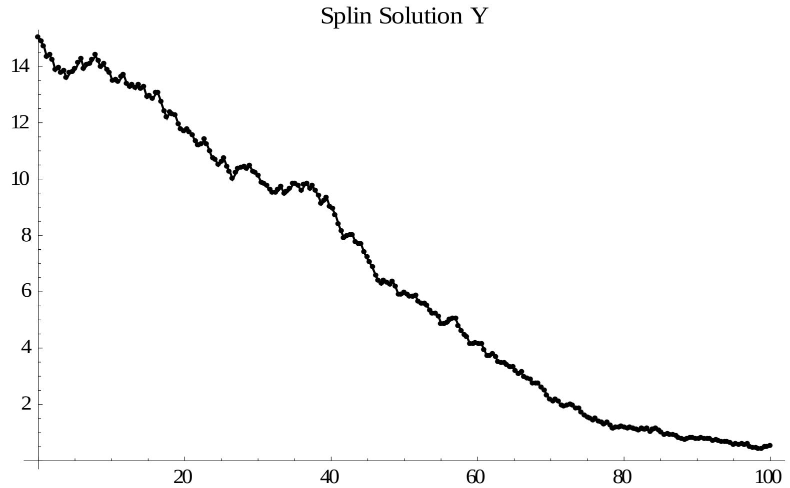

Fig. (7) Spline solution for the second population type Y, for $N = 2^7$, $h = 25/32$.

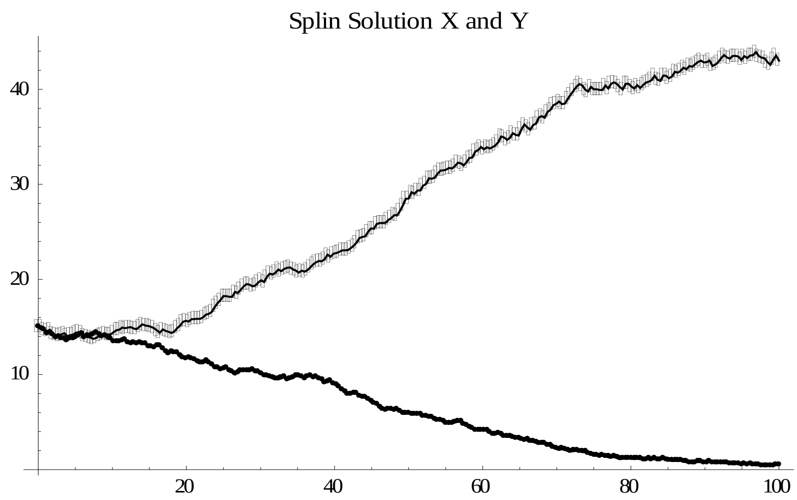

Fig. (8) Spline simulation of both X and Y types, for $N = 2^7$, $h = 25/32$.

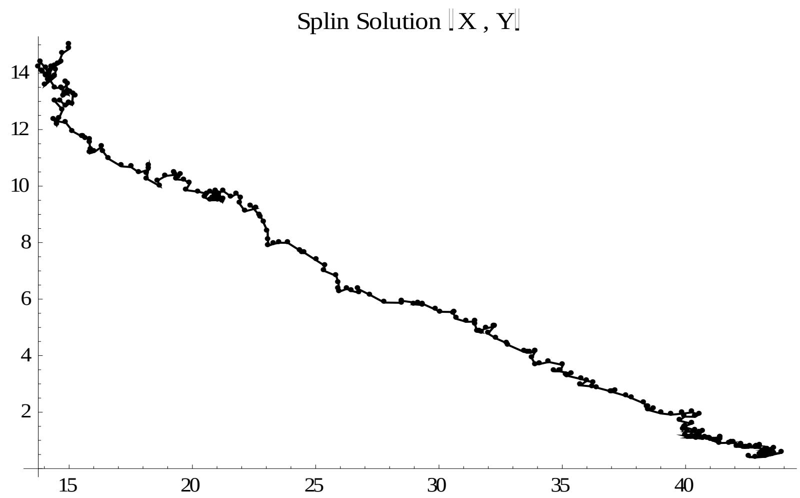

Fig. (9) spline simulation of X-Y plane, for $N = 2^7$, $h = 25/32$ in [0,100].

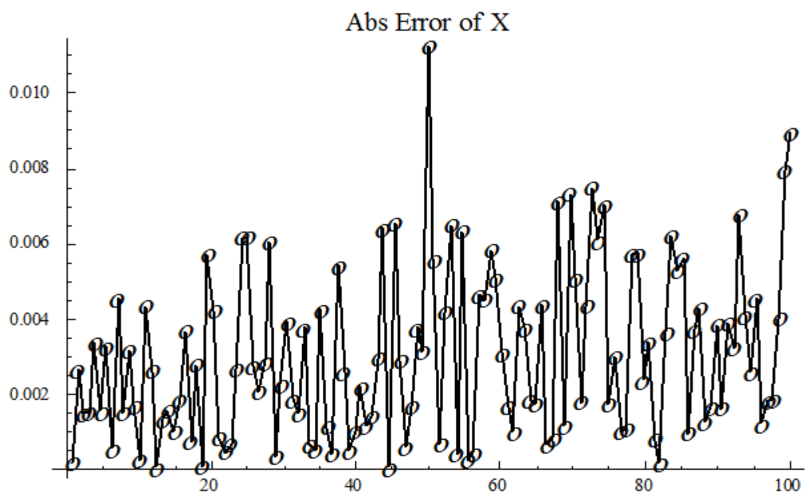

Fig. (10) The mean absolute error of the X-type spline solution for $h = 25/32$ in [0,100].

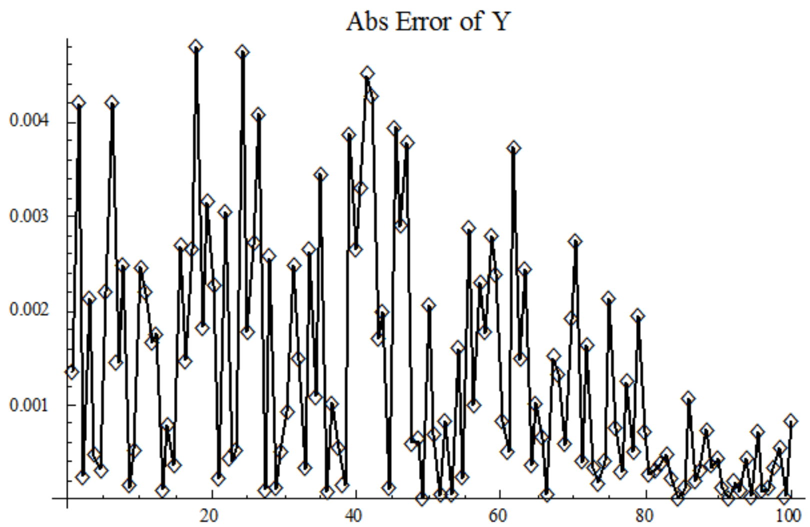

Fig. (11) The mean absolute error of the X-type spline solution for $h = 25/32$ in [0,100].

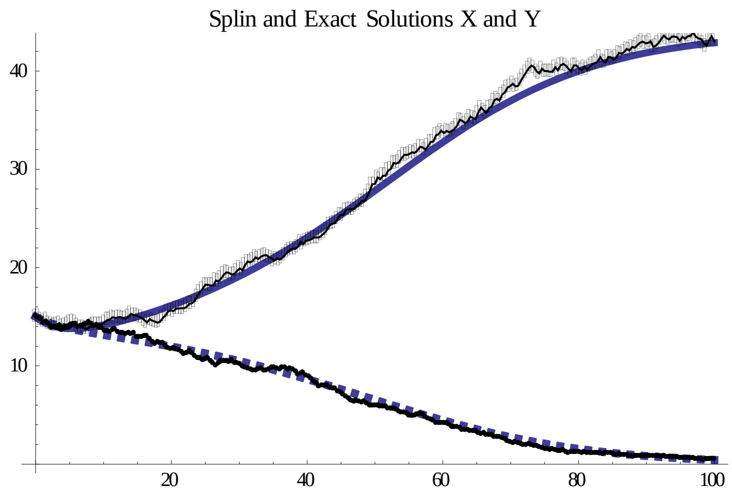

Fig. (12) Comparison of the spline simulation by our method with exact solution of X, Y of Problem 2 for N = 2 ^ 7, $h = 25/32$.

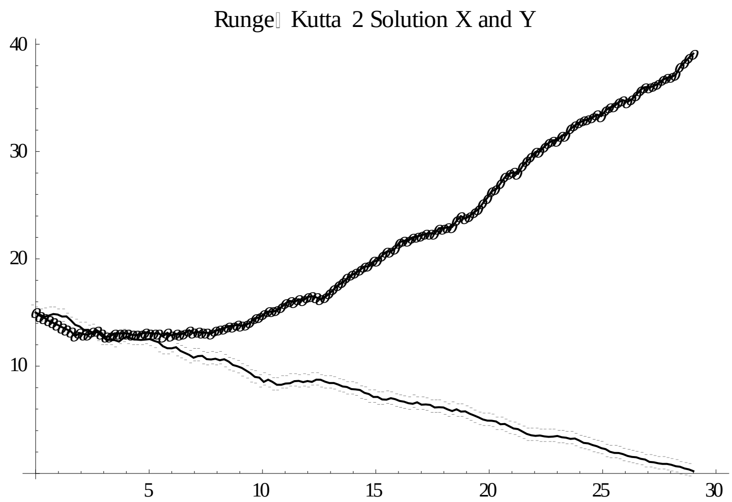

Fig. (13) The numerical solution by Runge Kutta method of Problem 2 with $h = 25/32$ in [0,30].

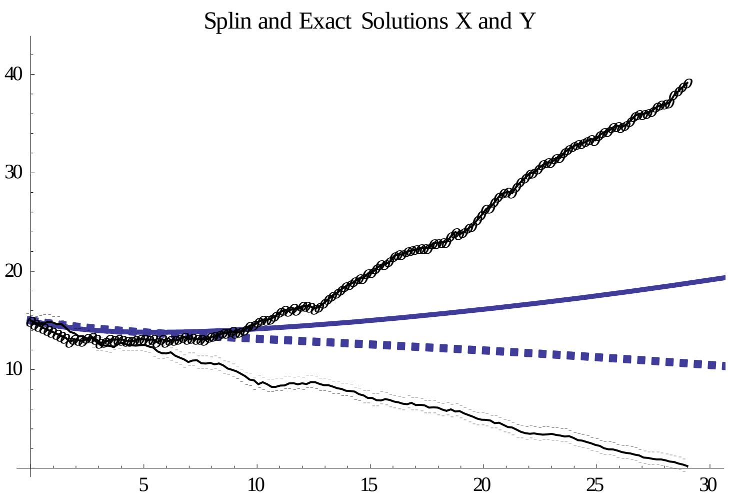

Fig. (14) Comparison of the numerical solution by Runge Kutta method with exact solution of *X*, *Y* of Problem 2 with, *h* = 25/32 in [0,30].

Notations: As a result of solving Problem 2, we notice from Figs 6-8 that the first population type *X* grows and flourishes while the second type, *Y* dies out and becomes extinct. Table 2 shows that the first population type *X* grows and increases, and when t = 87.5, the population exceeds 42 with mean absolute error, it is not more than 0.01, while the second population type *Y* is likely to start extinction, when at time t = 87.5, the population decreases to 0.827401, with an mean absolute error equal to approximately 0.004, and this suggests either the absence of any person or the presence of one person at most. To compare our proposed spline method with the Rung-Kutta method of second order [2], we note in Fig. 12 that our method simulates an exact solution to a very large extent from its beginning at $t = 0$ to its end at $t = 100$, while the numerical simulation using the Rung-Kutta method did not reach the end of the solution and only reached $t = 28$ approximately, and it failed completely after that, as is evident in Fig. 13. On the other hand, Fig.14 shows the divergence of the numerical solution from the exact solution by the Runge-Kutta method to a large extent and thus the inability of this method to solve such problems.

Problem 3 [11]

Consider the following stochastic equation systems:

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} X_t \\ Y_t \end{pmatrix}dt + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\begin{pmatrix} X_t\, dW_{1,t} \\ Y_t\, dW_{2,t} \end{pmatrix}, \quad t \in [0,2]$$

where the exact solution is given by

$$X(t) = X_0 \exp[(a_1 - \frac{1}{2}b_1^2)t + b_1 W_{1,t}], \; Y(t) = Y_0 \exp[(a_2 - \frac{1}{2}b_2^2)t + b_2 W_{2,t}].$$

Our technique is applied to solve this problem by choosing $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 1,$ $X_0 = 1, Y_0 = 2$ and T=2. We summarize in Table 3 spline solutions of *X*, *Y* and the absolute errors by our method for h=0.015625. In Fig.15-16 are plotted discretized Wiener process paths $W_{1,i}$ and $W_{2,i}$, for

N=128. In Fig.17-20, the spline solution with exact solution of X, Y and the absolute errors are plotted by our method for N= 128.

Table (3) The absolute error of Problem 3 by our technique for h=0.015625, in [0,2].

t_k	Absolute Error of X	Absolute Error of Y	Solution Spline X	Solution Spline Y
0.125	0.000311715	0.000388923	1.09510691828	2.687403726853
0.25	7.66351E-6	0.000589369	0.9180893240681	3.160085971482
0.375	0.000228364	0.000211111	0.5517547354758	3.144583677972
0.5	0.000206802	0.000192638	0.4220985401796	3.5314143667242
0.625	0.00022566	0.000587206	0.1818453653124	2.9755365896614
0.75	0.000181033	0.000990974	0.0941543431987	2.7487205058159
0.875	0.000165342	0.00100775	0.0827310462057	3.308091419314
1.0	0.000113249	0.00144636	0.0424617989635	3.042599099767
1.125	0.0000872678	0.00173041	0.0269440375661	3.111437393041
1.25	0.0000805738	0.00188729	0.0232726707386	3.7124195227918
1.375	0.0000698558	0.00209339	0.0179580653648	4.1863976636582
1.5	0.0000558092	0.00239371	0.0127350690205	4.525745781895
1.625	0.000039456	0.00279965	0.0078661309381	4.566064861148
1.75	0.0000235546	0.00329813	0.0042141739197	4.2907552225622
1.875	0.0000341351	0.00315654	0.0067132561969	6.9533030930411
2.0	0.0000411178	0.00243028	0.0084866098119	10.037500178815

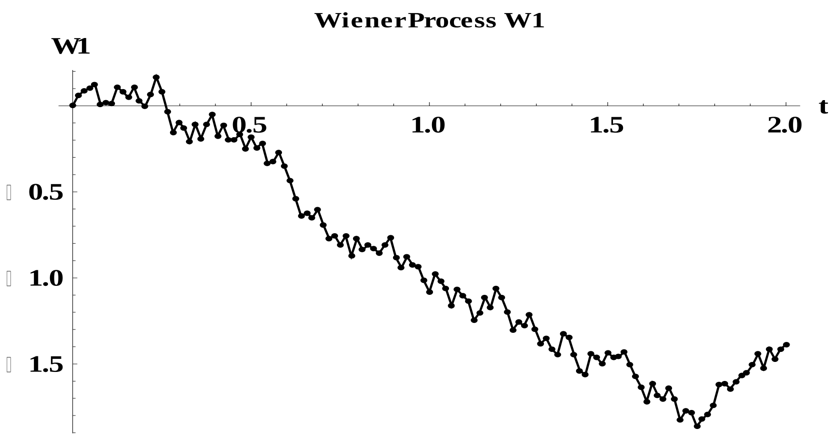

Fig. (15) Discretized Wiener Process path $W_{1,i}$, for N=128.

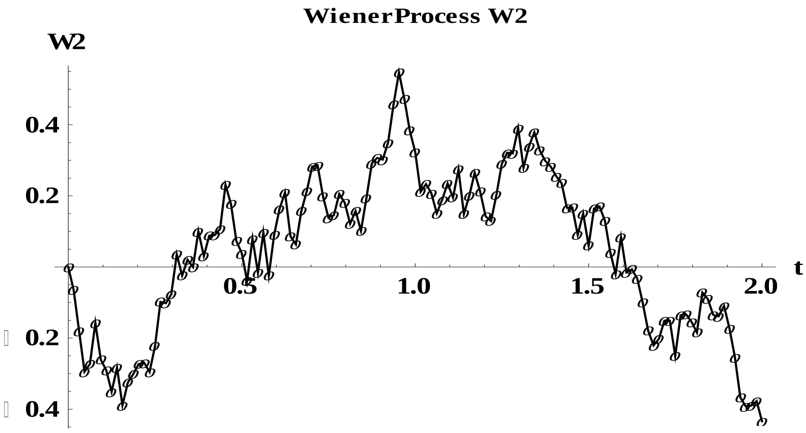

Fig. (16) Discretized Wiener Process path $W_{2,i}$, for N=128.

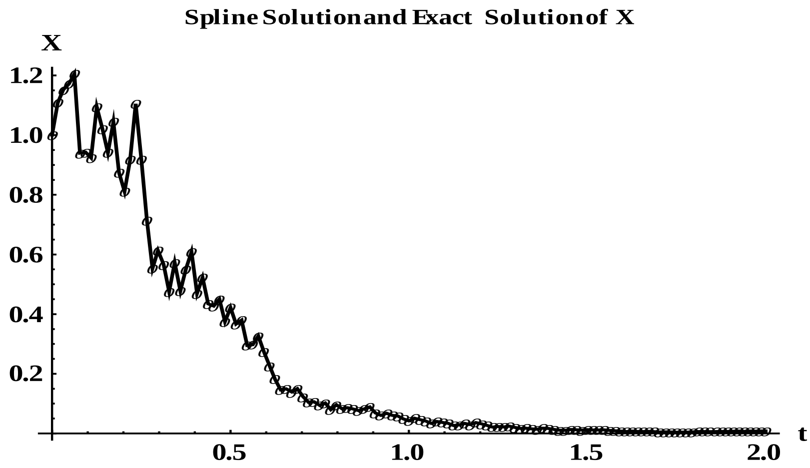

Fig. (17)The spline solution ••• with the exact solution ▁ for N= 128.

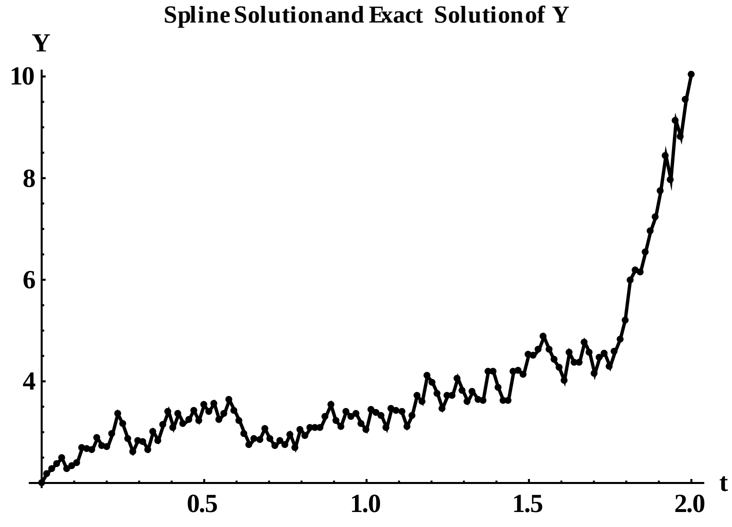

Fig. (18) spline solution ••• with the exact solution ▁ for N= 128.

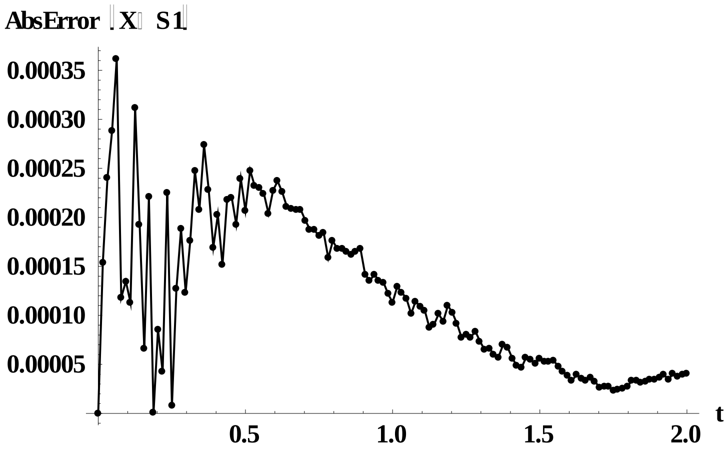

Fig. (19) Mean absolute error in the proposed *X*-spline solution with *h* =0.015625.

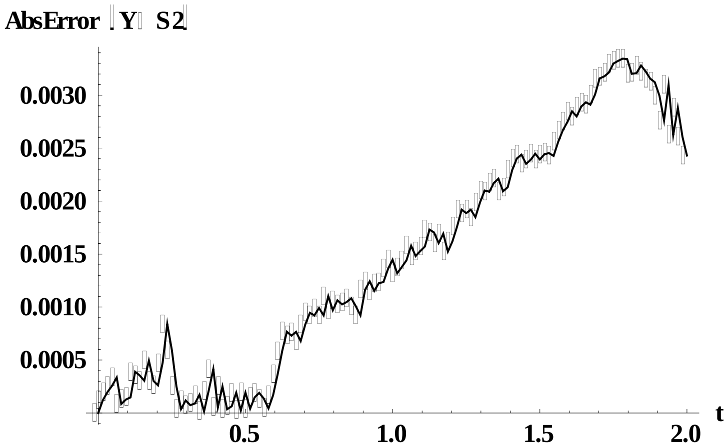

Fig. (20) Mean absolute error in the proposed *Y*-spline solution with *h*=0.015625.

6. Conclusion.

A two-point spline collocation method is presented for the numerical solutions of systems of stochastic differential equations in both linear and nonlinear states. The analysis of mean-square stability and convergence showed that our proposed spline method, when applied to a test model of stochastic differential equations systems, was mean-square stable and convergent of the third order. Fig's 2-20, and the results of calculations of mean absolute errors and the order of numerical convergence included in Tables 1-3 show that our method succeeded in simulating and matching the solution to a large extent in the linear and nonlinear cases of the three tested problems.

Given the success of the proposed technique in simulating the solutions of some stochastic differential equations systems, we recommend the following:

- Development of similar spline techniques for simulating the solution of systems of neutral stochastic differential equations.
- Study of mean-square stability analysis of spline collocation techniques for stochastic differential-algebraic systems.

References.

1- Haghighi & A. Rößler., Split-step double balanced approximation methods for stiff stochastic differential equations, International Journal of Computer Mathematics, 2018, pp.1-19.

2- Rathinasamy & K. Balachandran, Mean-square stability of second-order Runge–Kutta methods for multi-dimensional linear stochastic differential systems, Journal of Computational and Applied Mathematics, 219(1), 2008, pp.170 – 197.

3- Tocino & M.J. Senosiain., Mean-square stability analysis of numerical schemes for stochastic differential systems, Journal of Computational and Applied Mathematics, 236(10), 2012, pp.2660–2672.

4- E. Buckwar And C. Kelly, Towards a systematic linear stability analysis of numerical methods for systems of stochastic differential equations, SIAM J. Number. Anal, 48(1), 2010, pp 298–321.

5- K. Debrabant and A. Rößler, Classification of stochastic Runge–Kutta methods for the weak approximation of stochastic differential equations, Mathematics and Computers in Simulation, 77(4), 2008, pp.408 – 420.

6- L. J.S. Allen and E. J. Allen, A comparison of three different stochastic population models with regard to persistence time, Theoretical Population Biology, 64, 2003, pp. 439–449.

7- M. Baccouch, B. Johnson, A high-order discontinuous Galerkin method for Itô stochastic ordinary differential equations, Journal of Computational and Applied Mathematics, 308, 2016, pp. 138–165.

8- M. Bayram, T. Partal and G. O. Buyukoz, Numerical methods for simulation of stochastic differential equations, Advances in Difference Equations, (17), 2018, pp.1 – 10.

9- P. WANG, Three-stage stochastic Runge-Kutta methods for stochastic differential equations, Journal of Computational and Applied Mathematics, 222(2), 2008, pp.324-332.

10- Th. Famelis, F. Xanthos and G. Papageorgiou, Numerical solution of stochastic differential equations with additive noise by Runge–Kutta methods, European Society of Computational Methods in Sciences and Engineering, Vol. 4, No. 3-4, 2009, pp. 171-180.

11- M. Baccouch, H. Temimi, M. Ben-Romdhane, A discontinuous Galerkin method for systems of stochastic differential equations with applications to population biology, finance, and physics, Journal of Computational and Applied Mathematics, 11,pp.1-26,2020.