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Some Properties of Final Structured Spaces

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الملخص: الهدف من هذا البحث هو استكشاف بعض الخواص للبنية التفاضلية النهائية و برهنة نظريات متعلقة بها. فكما نعلم فان الفضاءات البنيوية هي عبارة عن تعميم لمفهوم متعددات الشعب الملساء. فاذا كان **{M,τ,C}** فضاء بنيوي حسب تعريف موستوف و **f** اي دالة فانه توجد بنية تفاضلية وحيدة معرفة علي الفضاء التبولوجي **N** بالنسبة للدالة **f** و تسمي بنية تفاضلية نهائية؛ و هي اقوي بنية تفاضلية تجعل الدالة **f** ملساء. عدد من البراهين الرياضية حول هذا النوع من الفضاء قد تضمنها هذا البحث، وقد تم دراسة عدد من الخواص كتركيب دالتين لهما بنية تفاضلية نهائية و العلاقة بين الفضاء النهائي و فضاء القسمة لنفس هذا الفضاء و دراسة الحالة التي تكون فيها الدالة **f** تقابلية. و قد تم التوصل الي ان تركيب دالتين لهما بنية تفاضلية نهائية ينتج دالة لها بنية تفاضلية نهائية و ان الفضاء البنيوي **N** يكون فضاء نهائي اذا وفقط اذا كان الفضاء **N/R** هو فضاء قسمة بنيوي. واخيرا تم توضيح العلاقة بين خاصية ان الفضاء **N** نهائي وبين كون الدالة **f** تشاكلا تقابليا بين فضاءات بنيوية (ديفيومورفيزوم)، حيث تم برهنة انه للدالة التقابلية **f: (M,τ,C) ⟶ N** يكون **N,D** فضاء بنيوي نهائي اذا و فقط اذا كانت الدالة **f** هي عبارة عن تشاكل تقابلي بين فضاءات بنيوية (ديفيومورفيزوم)،

الكلمات المفتاحية: فضاء بنيوي، دالة ملساء، بنية تفاضلية، فضاء نهائي، دالة تقابلية

Abstract: The aim of the study was to exp;ore several properties of special kind of morphisms between structured spaces called identification mappings of structured spaces and prove some results related to them. Let (M,τ,C) be a structured space in the sense of Mostow and let $f: (M,\tau,C) \longrightarrow N$, where N is arbitrary, be a function. There is a unique differential structure on N determined by f called the final, or identification, differential structure, and the space N then called the final structured space; this structure is greater than every differential structure on N such that f is smooth. Methodology: we provided mathematical proofs of several theorems related to final structured spaces. We investigate the composition of two functions have final differential structures, the relation between the final structured space and its quotient space, and the bijective mappings of structured spaces. Study results to the following: the composition of two identification mappings of structured spaces is also an identification mapping of structured spaces, a structured space N is a final structured space whenever N/R is a quotient structured space , a bijection f is an identification mapping of structured spaces if and only if the mapping f is a diffeomorphism.

Conclusion: The study has shown some properties of final structured spaces and quotient structured spaces. Moreover, the case when the identification mapping of structured spaces f is bijection is also investigated, it has been shown that the notions of diffeomorphisms and identification bijections of structured spaces are related.

Keywords: Structured Space, Smooth map, Differential Structure, Identification space, Diffeomorphism, Bijection

1. Introduction

Structured spaces or differential spaces are a generalization of the concept of smooth manifolds [1], [2]. A structured space in the sense of Mostow is defined to be a topological space with a sheaf of continuous real-valued functions which are closed with respect to composition with smooth Euclidean functions [1], [2].

Let (M, τ, C) be a structured space and let $f : (M, \tau, C) \rightarrow N$ be a function, where N is an arbitrary topological space. There is a unique differential structure on N determined by the function f called the final, or identification, differential structure; this structure on N with respect to f always exists and it is greater than every differential structure on N such that f is smooth [3].

In this paper, we will study some properties of these final structured spaces and prove some results related to them. Some properties of quotient structured spaces, which is a special kind of final spaces, are also proven. The case when the map f is a bijection is studied; and we show that the notions of diffeomorphisms and identification bijections of structured spaces are related.

2. Preliminaries

Differential structures, in the sense of Mostow, are defined as following.

Definition 2.1[1], [2] Let M be a topological space with a topology τ. A sheaf C of real continuous functions on M is said to be a differential structure (or a structural sheaf) on M if it satisfies the following condition : for any nonempty set $U \in \tau$, any sections $f_1, \ldots, f_n \in C(U)$, where $n \in \mathbb{N}$, and any function $\omega : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^∞, the composition $\omega \circ (f_1, \ldots, f_n)$ belongs to $C(U)$. The ordered pair (M, C), or the triple (M, τ, C), is called a structured space.

Definition 2.2. [1],[2] Let (M, C) and (N, D) be structured spaces. A continuous mapping $h: M \longrightarrow N$ is said to be smooth provided $g \circ h \in C(h^{-1}(U))$ for every section $g \in D(U)$.

Definition2.3 [4] Let Let (M, C) and (N, D) be structured spaces. A one-to-one mapping $h: M \longrightarrow N$ is said to be a diffeomorphism of (M, C) onto (N, D) provided both mappings $h: M \longrightarrow N$ and $h^{-1} N \rightarrow M$ are smooth. Then (M, C) and (N, D) are said to be diffeomorphic.

T. Bulati and A. M. A Ahmed [3] defined final structured space as following:

Definition 2.4. Let $\{(M_\alpha, \tau_\alpha, C_\alpha)\}$ be a collection of structured spaces and $\{f_\alpha : M_\alpha \rightarrow N\}$ be a collection of functions. Let τ_f be the final topology on N with respect to $\{f_\alpha\}$. A differential structure D on (N, τ_f) is said to be final with respect to the functions $\{f_\alpha\}$ if, for any structured space (K, F) and function $h : (N, D) \rightarrow (K, F)$, we have h is smooth if and only if $h \circ f_\alpha : (M_\alpha, C_\alpha) \rightarrow (K, F)$ is smooth for each α. In this case, (N, τ_f, D), or (N, D), is called the final structured space with respect to $\{f_\alpha\}$

Let $\{(M_\alpha, C_\alpha)\}$ be a collection of structured spaces in the sense of Mostow and $\{f_\alpha : M_\alpha \rightarrow N\}$ be a collection of functions. The following theorem from [3] shows that the final differential structure on

N with respect to $\{f_\alpha\}$ always exists and it is greater than every differential structure on (N, τ_f) such that each f_α is smooth.

Theorem 2.5. [3] Let $\{(M_\alpha, C_\alpha)\}$ be a collection of structured spaces and $\{f_\alpha : M_\alpha \to N\}$ be a collection of functions. Then the final differential structure D on (N, τ_f) with respect to $\{f_\alpha\}$ exists and is characterized by the following condition : if $U \in \tau_f$, then $h \in D(U)$ if and only if $h \circ f_\alpha \in C_\alpha(f_\alpha^{-1}(U))$ for each α. We concentrate attention on the case of a single function $f : M \to N$, so we have the following definition.

Definition 2.7. [3] Let $f : (M, \tau', C) \to (N, \tau, D)$ be any function. We say that f is an identification mapping of structured spaces if f is a surjection, $\tau = \tau_f$, and D is the final differential structure on N with respect to f. This differential structure on (N, τ_f) is also called the identification differential structure with respect to f, and we say (N, τ_f, D), or (N, D), is the identification structured space with respect to f. This identification differential structure on (N, τ_f) with respect to f is characterized as

$$D(U) = \{h : U \to \mathbb{R} : h \circ f \in C(f^{-1}(U))\}.$$

3. Some Properties of Identification Spaces:

In this section we prove some results related to final or identification differential structures.

Theorem 3.1. Let $f_1 : (M, C) \longrightarrow (N, D)$ and $f_2 : (N, D) \longrightarrow K$ be surjections, where D is the identification differential structure on N with respect to f_1. Then the identification differential structures on K with respect to f_2 and with respect to $f_2 \circ f_1$ coincide.

Proof. First, the proof that $\tau_{f_2} = \tau_{f_2 \circ f_1}$ can be found in [5]. Now, let F_1 and F_2 be the final differential structures on K with respect to f_2 and $f_2 \circ f_1$ respectively, and let $h \in F_1(U)$, So, we have

$$h \in \mathcal{F}_1(U) \Leftrightarrow h \circ f_2 \in D(f_2^{-1}(U))$$
$$\Leftrightarrow (h \circ f_2) \circ f_1 \in C(f_1^{-1}(f_2^{-1}(U)),$$

(Since D is the final structure on N with respect to f_1)

$$\Leftrightarrow h \circ (f_2 \circ f_1) \in C((f_2 \circ f_1)^{-1}(U))$$
$$\Leftrightarrow h \in F_2(U)$$

(Since F_2 is the final structure on K with respect to $f_2 \circ f_1$).

Consequently, $F_1 = F_2$.

Theorem 3.2. Let $f : (M, \tau, C) \to (N, \tau', D)$ be a smooth surjection. If there is a smooth mapping $\psi : (N, \tau', D) \to (M, \tau, C)$ such that $f \circ \psi = I_N$, then f is an identification mapping of structured spaces.

Proof. First, from [5] we have the proof that $\tau' = \tau_f$. Let $h \circ f \in C(f^{-1}(U))$ for some $h : N \to R$ and $U \in \tau'$. Since ψ is smooth, , then $(h \circ f) \circ \psi \in D(\psi^{-1}(f^{-1}(U)))$, that is, $h \circ (f \circ \psi) \in D((f \circ \psi)^{-1}(U))$. It follows that $h \in D(U)$, since $f \circ \psi = I_N$. Therefore, f is an identification mapping of structured spaces.

4 . Quotient Structured Spaces

Let (M, τ, C) be any structured space and let $\rho \subseteq M \times M$ be an equivalence relation in (M, C). Let us consider the quotient space $(M / \rho, \tau / \rho)$ and the sheaf C / ρ given by

$(C/\rho)(V) = \{f : V \to \mathbb{R} : f \circ P_\rho \in C(P_\rho^{-1}(V))\}$,

for $V \in \tau/\rho$, where $P_\rho : M \to M/\rho$ is the canonical projection of the point p onto its equivalent class [6]. The sheaf C/ρ is the final differential structure on $(M/\rho, \tau/\rho)$ with respect to P_ρ, and $(M/\rho, \tau/\rho, C/\rho)$ is called the quotient structured space ([6]).

Theorem 4.1. Let $(M/\rho, \tau/\rho, C/\rho)$ be a quotient structured space and let $P_\rho : M \to M/\rho$ be the projection. If $f : (M, C) \to (N, D)$ is a smooth map and if fP_ρ^{-1} is a single-valued, then $fP_\rho^{-1} : (M/\rho, C/\rho) \to (N, D)$ is smooth.

Proof. Since fP_ρ^{-1} is a single-valued then for each $x \in M$ we have

$$\begin{aligned} f(x) &= \left(f\left(P_\rho^{-1}P_\rho\right)\right)(x) \\ &= ((fP_\rho^{-1})P_\rho)(x) \end{aligned}$$

Therefore, we have

$$f = (fP_\rho^{-1})P_\rho$$

The smoothness of the map f implies that $(fP_\rho^{-1})P_\rho$ is smooth. Therefore, since P_ρ is an identification mapping of structured spaces, then we have fP_ρ^{-1} is smooth (from Definition 2.4).

Now, we have the following result.

Theorem 4.2. Let $f : (M, C) \to (N, D)$ be a relation-preserving smooth surjection. Let R_1 and R_2 be any relations on M, N, respectively. Then the mapping $\psi : (M/R_1, C/R_1) \longrightarrow (N/R_2, D/R_2)$ defined by $[x]_{R_1} \to [f(x)]_{R_2}$ is also smooth. Furthermore, ψ is an identification mapping of structured spaces whenever f is an identification mapping of structured spaces.

Proof. We have the commutative diagram [5]

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ P_2 \downarrow & & \downarrow P_1 \\ M/R_1 & \xrightarrow[\psi]{} & N/R_2 , \end{array}$$

where P_i is defined by $P_i(x) = [x]_{R_i}, i = 1,2$. Now, the proof that the mapping ψ is continuous and that the topology $\tau/_{R_2}$ is final whenever τ is final can be found in [5].

Let $h \in (D/R_2)(U)$ for some $U \in \tau/_{R_2}$, then $h \circ P_2 \in D(P_2^{-1}(U))$. The smoothness of the map f implies that

$$h \circ (P_2 \circ f) = (h \circ P_2) \circ f \in C(f^{-1}(P_2^{-1}(U))) = C((P_2 \circ f)^{-1}(U)))$$

Therefore, since $P_2 \circ f = \psi \circ P_1$ we have

$(h \circ \psi) \circ P_1 \in C((P_1^{-1}(\psi^{-1}(U)))$.

Thus $h \circ \psi \in (C/R_1)(\psi^{-1}(U))$ and the mapping ψ is smooth.

Now, suppose that D is the identification differential structure on N with respect to f; then Theorem 2.1. shows first that $\psi \circ P_1 (= P_2 \circ f)$ is an identification mapping of structured spaces and then that ψ is also.

5. Identification Bijections of Structured Spaces

Let $f: (M, \tau, C) \to (N, \tau', D)$ be a identification mapping of structured spaces where f is a bijective map. In this section, we shall show that the notions of diffeomorphisms and identification bijections of structured spaces are related.

First, it is important to note that a smooth bijection need not be a diffeomorphism, the following example shows that.

Example 5.1. let (M, τ, C) be a structured space. Then for any differential structure D defined on (M, τ) and satisfied that $D(U) \subset C(U)$ and $C \neq D$ for each nonempty set $U \in \tau$, the identity map $I_M : (M, \tau, C) \longrightarrow (M, \tau, D)$ is a smooth bijection, but I_M is not a diffeomorphism.

Now, we have the following result which shows the relation between diffeomorphisms and identification bijections of structured spaces.

Theorem 5.2. Let $f: (M, \tau, C) \to (N, \tau_f, D)$ be a bijection. Then f is an identification mapping of structured spaces if and only if the mapping f is a diffeomorphism.

Proof. Suppose that f is an identification mapping of structured spaces, so immediately we have f is smooth. Now we prove that f^{-1} is also smooth. Obviously, τ is the final topology on M with respect to f^{-1} (see [5],[7]). Let $h \in C(U)$ for some $U \in \tau$, that is, $(h \circ f^{-1}) \circ f \in C(f^{-1}(f(U)))$. Since D is the identification differential structure on N with respect to f, we have $h \circ f^{-1} \in D(f(U)) = D((f^{-1})^{-1}(U))$. Hence, the mapping $f^{-1}: (N, D) \to (M, C)$ is smooth. Consequently, the mapping f is a diffeomorphism.

Conversely, suppose that the mapping f is a diffeomorphism. Thus, the mappings $f: M \to N$ and $f^{-1}: N \to M$ are smooth. Since $f \circ f^{-1} = I_N$, then, by Theorem 3.2, f is an identification mapping of structured spaces.

Theorem 5.3. Let $f:(M,\tau,C) \to (N,\tau_f,D)$ be a bijection, where D is the identification differential structure on N with respect to f. Then the mapping $f^{-1}:(N,\tau_f,D) \to (M,\tau,C)$ is an identification mapping of structured spaces.

Proof. Obviously, τ is the final topology on M with respect to f^{-1} (see [5],[7]). By Theorem 5.2, the mapping f^{-1} is smooth. Now, let $h \circ f^{-1} \in D\big((f^{-1})^{-1}(U)\big) = D(f(U))$ for some $h: U \to R$ and $U \in \tau$. Since D is the final differential structure on N with respect to f, then

$$h = (h \circ f^{-1}) \circ f \in C\Big(f^{-1}\big(f(U)\big)\Big) = C(U).$$

This shows that C is the final differential structure on M with respect to f^{-1}.

6. Conclusion

The study has shown several properties of final structured spaces. Moreover, the case when the map f is bijection is investigated.

Interesting future studies and more properties can be investigated. Final mapping of structured spaces might be studied for special maps or special topological spaces. Structured subspaces might also be studied to show in which cases these subspaces are also final.

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