Journal of Natural Sciences,
Life and Applied Sciences
Issue (2) 1 - June 2017
ISSN: 2518-5780

المجلة العربية للعلوم و نشر الأبحاث
Arab Journal of Sciences & Research Publishing
AJSRP

Maximum and Minimum Energy in Vortex Motion

Amna Mohammed Osman Elzebair

University of El –Neleen- Sudn

Abstract: This paper is discussion the Maximum and minimum energy in vortex motion exploited the conservation of vorticity as well as energy in two-dimensional motion of an incompressible inviscid fluid, to find some quite general criteria for stability of steady basic flows, and also steady Bènard convection sets in at the critical value of the Rayleigh number. This paper also discussion some experiments with a silicone oil (of Prandtl number100). (Abstract text Times New Roman, size 12, italic. Spacing-SINGLE). These guidelines provide instructions to format your paper. Please write directly into the template or copy your finished text into it choosing 'match destination formatting'. Please use the predefined formatting styles instead of applying your individual settings. The paper shall be written in compliance with these instructions. Please review this document to learn about the formatting of text, table captions and references. The conference proceedings will be published in an electronic format. The Abstract should be no more than 200 words and one paragraph only. Avoid quotation and citing references in your abstract.

Keywords: Landau constant- Nusselt number - Prandtl number -Rayleigh number - Reynolds number.

1. Introduction

Hydrodynamic stability has a lot in common with stability in many other fields, such as magneto hydrodynamics plasma physics, elasticity, rheology, combustion and general relativity. The physics may be very different but the mathematical essence is that the physics is modelled by nonlinear partial differential equations and the stability of known steady and unsteady solutions is examined . Hydrodynamics happens to be a mature subject (the Navier-Stokes equations having been discovered in the first half of the nineteenth century), and a given motion of a fluid is often not difficult to produce and to see in a laboratory, so hydrodynamic stability has much to tell us as a prototype of nonlinear physics in a wider context .

The kinetic energy of a flow of an incompressible inviscid fluid has a stationary value when the flow is steady, Kelvien (1887) recognized that the flow is stable if the stationary value is either a maximum or a minimum. Arnol'd (1966) proved these results by use of the calculus of variations and applied them to hydrodynamic stability of various flows.

2. Maximum and minimum energy in vortex motion

2.1. Conservation of Vorticity

Arnol'd (1965a) exploited the conservation of vorticity as well as energy in two-dimensional motion of an incompressible inviscid fluid, to find some quite general criteria for stability of steady basic flows. He considered first a general flow in a doubly connected plane domain D with fixed boundary Γ which consists of two smooth closed curves Γ_1 and Γ_2. This flow is governed by the vorticity equation

$$\frac{\partial \Delta \psi}{\partial t} = \frac{\partial (\psi, \Delta \psi)}{\partial (x, z)} \qquad (3.1.1)$$

in the xz-plane, where $u = \partial \psi / \partial z$ and $w = -\partial \psi / \partial x$. The boundary condition on the streamfunction ψ may be expressed so that $\psi = 0$ on Γ_1 and $\psi = q(t)$ on Γ_2 for some function q. It follows that the energy integral $\mathcal{K} = \frac{1}{2} \iint_D (\nabla \psi)^2 \, dx \, dz$ is a constant of the motion. Also the vorticity $\Delta \psi$ of each incompressible fluid particle is constant, so the integral $\mathcal{F} = \iint_D f(\Delta \psi) \, dx \, dz$ is another constant of the motion for any integrable function f.

Arnol'd considered next the extreme of the functional **H** $= \mathcal{F} + \mathcal{K}$.
with respect to variations of smooth function ψ with satisfy the boundary conditions and also $\oint_{\Gamma_j} \partial \psi / \partial n \, ds = c_j$ for $j = 1, 2,$ where C_1 and C_2 are some constants.
The first variation is given by

$$\delta \mathcal{H} = - \iint_D \{ \nabla \psi \cdot \nabla \delta \psi + f'(\Delta \psi) \Delta \delta \psi \} \, dx \, dz$$

$$= - \iint_D \Delta h \, \delta \psi \, dx \, dz + \oint_{\Gamma} \left(\frac{\partial h}{\partial n} \delta \psi + f' \frac{\partial \delta \psi}{\partial n} \right) ds, \qquad (3.1.2)$$

on integrating by parts, where $h = \psi - f'(\Delta \psi)$. Therefore

$$\delta \mathcal{H} = - \iint_D \Delta h \, \delta \psi \, dx \, dz \qquad (3.1.3)$$

Similarly, the second variation can be shown to be

$$\delta^2 \mathcal{H} = \frac{1}{2}\iint_D \{(\nabla\delta\psi)^2 \quad f'(\psi)(\nabla\delta\psi)^2\}\, dx\, dz \qquad (3.1.4)$$

Now the stream function Ψ of any steady basic flow satisfies the equation

$$\Psi = g(\nabla\Psi) \qquad (3.1.5)$$

for some function g. This follows from the vorticity equation

(3.1.1) or from the variational equation (3.1.2) if $f' = g$ so that $h \equiv 0$. Therefore $\mathcal{H}$ is a local minimum or maximum where $\psi = \Psi$ if the integrand of equation (3.1.3) is positive or negative definite respectively at each point of D. Arnol'd deduced, by appeal to a generalization of a well-known result for dynamical systems of finite dimension, that the basic flow is stable to disturbances of finite (but possibly small) amplitude if the integrand of equation (3.1.3) is either positive or negative definite in some frame of coordinates. (The essence of Arnol'd's deduction can be understood by use of a three-dimensional analogy. Suppose then that $\mathcal{H}$ = constant is represented by some surface in $\mathbb{R}^3$ and K by a Cartesian coordinate. Then each fluid motion is represented by the trajectory where a surface $\mathcal{H}$ = constant is cut by a plane K = constant. If $\mathcal{H}$ has a maximum or a minimum when $\psi = \Psi$, then the surface $\mathcal{H} = \mathcal{H}_0$ is locally a paraboloid touching the

plane $\mathcal{H} = \mathcal{H}_0$ at the 'point' Ψ, where

$\mathcal{H}_0 = \mathcal{H}(\Psi)$ and $\mathcal{K}_0 = \mathcal{K}(\Psi)$. Therefore if ψ takes any value ψ_1 near to Ψ at some instant the ensuing motion will be represented by the elliptic trajectory where the plane $\mathcal{K} = \mathcal{K}(\psi_1)$ cuts the paraboloid $\mathcal{H} = \mathcal{H}(\psi_1)$. It follow that K will remain near K_0 and therefore that ψ will remain near Ψ for all time. The method depends upon finding a local maximum or minimum of $\mathcal{H}$, and so applies to disturbances of finite amplitude, but these disturbances may be of small amplitude if another extremum happens to be nearby .)

Arnol'd (1965a) applied this result first to parallel flow between the planes Γ_1, with equation $z = z_1$ and Γ_2 with $z = z_2$. In this example, the basic velocity is given by $U(z) = \Psi'(z) = dg(U')/dz = g'(U')U''$ and the basic vorticity by . He deduced that the flow is stable if $f'(\Delta\psi) = U/U''$ so that $\Delta\Psi = U'(z)$, for any $(U - U_0)/U'' \geq 0$. This result follows also if $z_1 \prec z \prec z_2$ for $U/U'' \geq 0$ constant U_0.

Application of boundary-layer theory to cellar instability

When steady Bènard convection sets in at the critical value of the Rayleigh number, the flow is very slow, but, as the Rayleigh number increases far above its critical value, a strong cellular flow develops. This experimental observation

led Pillow (1952) to assume that there is a strong steady cellular flow and to neglect viscosity and heat conduction in the interior of the cells. Thus he deduced that the fluid in the interior has uniform temperature and vorticity, and he analysed the boundary layers at the rigid plates and at the edges of the cells. This led to the result that the heat transfer is proportional to $(T_0 - T_1)^{5/4}$, where T_0 is the

temperature of the lower plate and T_1 of the upper plate, i.e. that the Nusselt number $Nu \propto Ra^{1/4}$. This agrees quite well with the values measured in many experiments when *Ra* is large enough for the convection to be strong yet not so large that the

flow is unsteady. For example , Threlfall (1975) found that $Nu = 0.173Ra^{0.28}$ very accurately when $4\times10^5 \prec Ra \prec 2\times10^9$ in some experiments on liquid helium (for which $\Pr \approx 0.8$).

Batchelor applied similar ideas to Taylor vortices, and deduced that the torque between the rotating cylinders over a length *H* is approximately proportional to

$$H\rho\Omega_1^2 R_1^4 (\nu/\Omega_1 R_1 d)^{1/2} (d/R_1)^{1/4}$$

when the outer cylinder is at rest, where the gap width $d = R_2 - R_1$, etc. Again there is fair agreement with observed values of the torque when the Taylor number is a few times its critical value (Donnelly & Simon 1960).

3. Some applications of the nonlinear theory

3.1. Bènard convection

Bènard convection of a Boussinesq fluid provides the simplest case for which bifurcation and transition to turbulence may be studied theoretically, yet a case for which it is not easy to compare many theoretical and experimental results, because of the complexity of the properties of real fluids. In reviewing the nonlinear theory, we here, first discuss convection of a Boussinesq fluid between two infinite horizontal planes, and then consider the modifications due to non- Boussinesq properties of fluids and to the presence of side walls. We shall not
emphasize the various boundary conditions, but often in the history of the development of the theory a particular nonlinear phenomenon has been analysed first for the simplest case of free boundaries and then for rigid boundaries, only quantitative differences between the effects of the boundary conditions being found.

Gor'kov (1957) and, independently but more completely, Malkus & Veronis (1958), found weakly nonlinear *steady* solutions with various plain-forms of the cells for slightlysupercritical values of the Rayleigh number R. Their work implies that the Landau constant is positive, and gives the magnitude and structure of the disturbance, and hence the heat transfer, as a function of R for each assumed plan-form, although it does not distinguish flows for which there is ascent at the centres of the cells from those for which there is descent.

In the classical linear theory of Bènard convection the most unstable mode is degenerate, in the sense that the horizontal plan-form of its cells is indeterminate, although its vertical variation and horizontal wavenumber are determinate; the ultimate plan-form of a given disturbance is thus determined by the initial horizontal distribution of the most unstable mode. Yet in experiments with a given apparatus, the plan-form of the cells may consistently be observed and appears to
be largely independent of the initial condition .Malkus & Veronis (1958)suggested that the observed plan-form should correspond to that one of all the possible nonlinear solutions which is itself stable with respect to infinitesimal disturbances of the form of the other solutions.

Schlüter, Lortz &Busse (1965) went on to consider the general study solution of a given horizontal wavenumber *a* for small positive value of $R - R_c$. They found an infinity of such
solutions, and showed that of these the only stable ones correspond to two- dimensional roll cells.

Of these roll cells, some with $a \succ a_c$ are stable but none $a \prec a_c$ is. Schlüter *et al*, also calculated the heat transfer for small values of $R - R_c$, showing that for convection between rigid boundaries the Nusselt number *Nu* satisfies $R_c(Nu - 1)/(R - R_c)$

$$\rightarrow \begin{cases} (0.69942 - 0.00472pr^{-1} + 0.00832pr^{-2})^{-1} & for\ rolls \\ (0.77890 + 0.03996pr^{-1} + 0.06363pr^{-2})^{-1} & for\ square\ cells \\ (0.89360 + 0.04959pr^{-1} + 0.06787pr^{-2})^{-1} & for\ hexagonal\ cells \end{cases}$$

As $R \downarrow R_c$ for fixed $pr \neq 0$. (4.1.1)

These results are consistent with Malkus's(1954a,b) hypothesis of maximal heat transport, namely that , of all the possible motions, the one realized will be that for which the heat transport is a maximum, because the Nusselt number is largest for rolls.

Using a Galerkin numerical method, Busse (1967a) extended these results of Schlüter *et al.* for values of R up to 30 000 for the special case of rigid boundaries and large values of the Prandtl number. His results are illustrated in Fig.4.1.1 There are no steady solutions of horizontal wavenumber a for $R \prec R_1(a)$, where R_1 here

denotes the least eginvalue of the appropriate linear problem .For $R \succ R_1$, there is an infinity of steady finite-amplitude solutions, of which the only stable ones are two-dimensional rolls for values of a and R within the region bounded by the curves (B) and (C). The curve (B) is the margin outside of which each roll becomes unstable to disturbances of the form of oblique rolls, which grow without oscillations. This is called *zigzag instability* on account of the pattern of the ensuing steady convection, which Busse &Whitehead (1971) observed in some experiments with a silicone oil (of Prandtl number100). The curve (C) of Fig.4.1.1. is the margin outside of which each two-dimensional roll becomes unstable to disturbances of the form of rolls perpendicular to itself? This is called *cross-roll instability* (Busse &Whitehead 1971), and it also sets in as a cellular motion with exchange of stabilities. This represents a limit of current theoretical knowledge. If, however, R is above the value (about 23000) of the topmost point of the curve (C) in Fig.4.1.1., then another form of instability has
been observed experimentally by Busse &Whitehead (1971) to set in; this instability leads to a
steady three-dimensional motion, as if composed of two perpendicular rolls of different wavenumbers, and is called *bimodal instability*. At

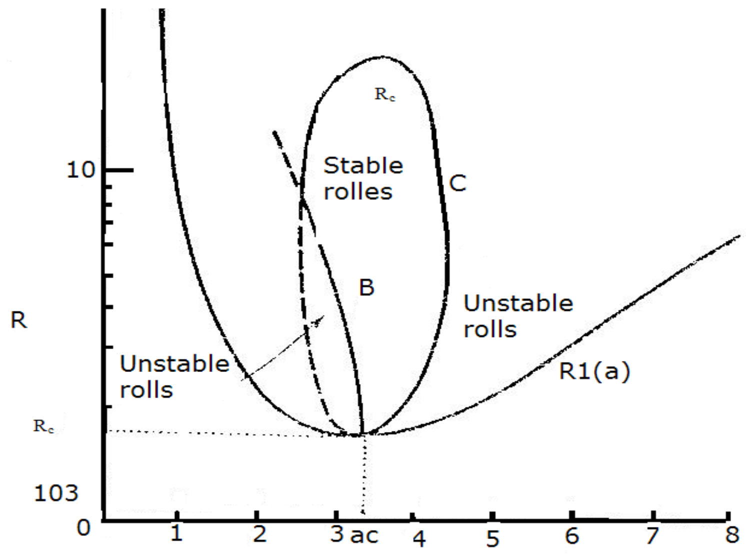

Fig.3.1.1 The region of stability of two-dimensional rolls in a fluid of infinite Prandtl number with respect to non-oscillatory perturbations. (After Busse 1967a.)

even higher values of the Rayleigh number a further instability is observed, the motion ceases to be steady, and an oscillatory flow arises.

These theoretical and experimental results differ somewhat if the value of the Prandtl number is not large; the quantitative results depend upon its value, though not strongly. Also when the Prandtl number is sufficiently low (less than about 1.1 for rigid and 3.5 for free boundaries) the curve (C) is the margin outside of which each roll becomes unstable to disturbances of the form of rolls parallel to it self; this is a special case of the side-band instability discovered by Eckhaus . Also , Busse & Clever (1979) showed both theoretically and experimentally that two further types of instability of rolls, called *knot instability* and *skewed varicos*

e instability , occur when the value of the Prandtl number lies in a certain finite positive interval. The instability of two-dimensional steady rolls in convection of a fluid of small Prandtl number was investigated theoretically by Busse (1972) and Clever & Clever (1974). They found that when the amplitude of a roll exceeds a certain critical value there is shear (rather than gravitational) instability of the roll. Thus an oscillatory three-dimensional flow sets in at much lower values of the Rayleigh number than it does for a fluid of large Prandtl number.

The occurrence of hexagonal cells is now attributed to non-Boussinesq properties of the fluid. Palm (1960) was the first to consider theoretically the effect of the variation of viscosity with temperature upon nonlinear cellular convection. Of the many subsequent theoretical papers

those by Busse (1967b) and Palm, Ellingsen & Qjevk (1967) may be chosen to give the current view. The kinematic viscocity is assumed to vary slightly with temperature. It is then shown at length that for $R_c \prec R \prec R_1$ only hexagonal cells are stable, for $R_1 \prec R \prec R_2$ both hexagonal cells and two-dimensional rolls are stable where here $R_1 - R_c$ and $R_2 - R_c$ are certain small functions of the parameters which are proportional to the derivative $d\nu/d\theta$ of the kinematic viscocity of the fluid with respect to the temperature. In addition, subcritical instability occurs; the steady hexagonal cells are in fact stable also for the small range $R_{-1} \prec R \prec R_c$, and there is transcritical bifurcation at $R = R_c$, where $R_c - R_{-1}$ is a certain function proportional to $d\nu/d\theta$. For each hexagonal convection cell the flow ascends at center if $dn/dq \text{ p } 0$ descends if $dn/dq \text{ f } 0$.

These theoretical results are confirmed qualitatively by the experiments of Hoard, Robertson & Acrivos (1970) which were done in a cylinder containing a liquid hydrocarbon (namely Aroclor 1248) whose viscosity varies strongly with temperature.

The side walls also have important effects, which must be borne in mind when comparing theoretical with experiments results. This effects of side walls are particularly important for convection in deep layers, experiments showing concentric circular rolls arise in a container with acircular cross-section and square cells in one with a square cross-section.

The early experimentalists carefully measured the heat transfer as a function of the Rayleigh number for various fluids. This determined the critical Rayleigh number in agreement with the linear theory. Now the heat transfer also depends, albeit less strongly, upon the plan-form, the Prandtl number and other factors. So it is difficult to compare theoretical and experimental results in detail. Nonetheless, there is in general satisfactory agreement between them. In particular, Koschmieder & Pallas (1974) measured the heat transfer across convecting layers (of depth $d \approx 5\ mm$) of various silicone oils (with 500∂ pr ∂1700) in a cylindrical container (of diameter D $\approx$ 13cm) for 0 < R ∂ 170 R_c. For small positive values ofthey saw concentric circular rolls and their measurements give $R_c - R$ this compares well with the theoretical result given by $R(Nu - 1)/(R - R_c) @ 1.48;$ equations (4.1.1), which gives $R_c(Nu - 1)/(R - R_c) \circledR\ 1.43$ for two-dimensional rolls as $R^{-} R_c$ for any large values of pr. Koschmieder & Pallas's measurements of Nu when R was a few times greater than its critical value agree quite well with the numerical results of Lipps & Somerville (1971). Ahers (1974) made some exceptionally accurate measurements on convection in layers (d ≈ 1mm) the experimental techniques available at low temperatures allowed him to work with very small temperature differences across the layers of fluid , conditions for which the

Boussinesq approximation is very good . He found that Nu increased from the value one smoothly as R increased through R_c , not abruptly at the onset of instability as given by the usual theory. This result is in accord with the theory of imperfections and may be due to side-wall effects . For slightly larger values of R , Ahlers fitted his data with the formula

$$Nu = 1 + 1.034e + 0.981e^3 - 0.866e^5 \qquad (4.1.2)$$

for $1.07R_c \le R \le 2.5R_c$ and pr = 1.17, where $e = (R - R_c)/R$. This is in qualitative agreement with the theoretical results given by equations (4.1.2), although Ahlers did not observe the plan-form. He also found that $Nu - 1 = 0.77\left\{\left(R/R_c\right) - 1\right\}^{0.334}$ for $30R_c \le R \le 150R_c$. Threlfall (1975), in further experiments on convection in helium at low temperatures, found that $Nu = 0.173R^{0.28}$ for $230R_c \le R \le 10^6 R_c$, in fair agreement with Pillow' s theoretical result. Observation of steady convection cells show that their size tends to increase, and therefore their horizontal wavenumber to decrease, as the Rayleigh number increases above its critical value. This seems to be in disagreement with the theoretical results illustrated in Fig. 4. 1.1, because the stable two-dimensional rolls are found to have wavenumbers greater than those on the boundary (B). This analytical conclusion is supported by numerical work. Resolution of this disagreement is awaited, but again it must be remembered that the experiments are not exactly modelled by the ideal assumptions of the theories without side effect, etc.

Krishnamurti (1968) considered both theoretically and experimentally how convection is initiated by slow heating of the lower plate or cooling of the upper plate. She assumed that the average basic temperature of a Boussinesq fluid increases or decreases slowly at a constant rate. Her results are similar to those for a steady basic state of a fluid whose viscosity depends upon the temperature. In particular, she found that there is subcritical instability and that hexagonal rather than roll cells arise. The fluid ascends or descends at the centers of the cells according to whether the average basic temperature decreases or increases respectively with time. The nature of the onset of turbulence itself remains obscure, and may indeed depend upon the fluid, the apparatus, and how the convection is initiated. Modern experimental techniques, however, make it possible to measure and analyses the frequency spectra of unsteady flows, and offer some evidence of aperiodicity like that of a strange attractor as well as of the periodicity and quasi-periodicity of the flows which we have described above. Ahlers (1974) observed that small-amplitude oscillations of the heat transfer about a mean value arose with a broad frequency band as R increases through a critical value of R_t equal to about twice R_c. In later experiments Ahlers & Behringer (1978) found that R_t decreases with aspect ratio D/2d of the cylindrical container and they suggested that $R_t \leftarrow R_c$ when D/2d=57. The periodiccomponent of the flow had not been observed before, perhaps because its time scale is slow. The nonlinear phenomena of only classical Bènard convection, rather than its many variations and application, have been described for brevity. Some of the in Hopfinger, Atten & Busse (1979), where several modern references may be found .

REFERENCES

1. Arnold, V. I. (1965a).Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid *.Dokl.Akad. Nauk SSSR* 162,975-8. Translated in *Soviet Math. Dokl. 6*, 7737(1965).[pp.432,434.]
2. Batchelor, T.B. (1962).T. fluid Mech.14,593. Cambridge University press.
3. Bénard, H (1900). Les tourbillons cellulites danseuse nappe liquid. *Revue Gén. Sci. Pur. Appl.*11, 1261-71 and 1309-28. [p. 32.]
4. Benjamin, T. B. (1972). The stability of solitary waves, Proc. Roy. Soc, A 328, 153-3.
5. Boussinesq, J. (1903) . *Théorie analytique due la chaleur*, vol.2, p. 172. Paris: Gauthier-Villars. [p. 35.]
6. Busse, F.H.& Whitehead, J.A (1971). Instabilities convection rolls in a high Prandtl number. J. Fluid Mech.47,305-20. . [p. 438.]
7. chandrasekhar,S. (1961). *Hydrodynamic and hydromagnetic stability* . Oxford: Clarendon Press. [pp. 25,28.49,62,64,66,67,79,93,98,99,103,110,121,205,327,340.]
8. Couette, M.(1890). Études sur le frottement des liquids. *Ann. Chim. Phys.* (6) 21,433-510. [p. 70.]
9. Davis , R.E (1969). On the high Reylonds number flow over a wavy boundary. J. *Fluid Mech.*36, 337-46. [p. 421.]
10. Diprima, R.C. (1954).A note on the asymptotic solutions of the equation of hydrodynamic stability *J. math. Phys.* 33, 249-57. [p. 177.]
11. Drazin, P.G.& W. H . Reid (1981) . Hydrodynamic Stability Cambridge University press.

12. Kelvin, Lord (1887). On the stability of steady and periodic fluid motion. Phil. Mag. (5) 23, 459-64, 529 – 39. Also *Mathematical and physical papers* (1910), vol. IV,pp. 166-83. Cambridge University press. [p. 432.]
13. Landau, L. D. & Lifshitz, E. M. (1959). *Fluid mechanics*. Vol. 6 of course of theoretical physics. (Translated from the Russian by J.B. Sykes and W.H. Reid.) London: Pergamon press. [pp. 1, 349, 370.]

الملخص

هذه الورقة تناقش الطاقة القصوي والدنيا في الحركة الدوامية لمائع (السائل والغاز) غير اللزج والغير مضغوط ، معتمدة علي حركتان هما : الحركة الدوامية والحركة ثنائية البعد للمائع و الخاصيتين هما للسائل غير اللزج والغير مضغوط فقط

وأيضا تناقش الورقة الحمل الحراري وعلاقته بثابت بينارد الذي يحدد القيمة الحرجة لعدد رينولدز وتتناول الدراسة ايضاً مناقشة بعض التجارب مع زيت السيليكون تحت تأثير رقم براندتل (100)

الكلمات المفتاحية :ثابت لانداو – رقم نيوسلت – رقم براندل – رقم ريلايت - عدد ريونولدز
